# **CSE 12**:

# Basic data structures and object-oriented design

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Lecture Sixteen
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# Sorting, continued

# Review of selection sort

- Pseudocode:

```
void selectionSort (int[] array) {
  While size of unsorted part > 0:
    Find largest element e of unsorted part
    Swap e with right-most element of unsorted part
}
```

- Time cost: $O(n^2)$ in worst, best, and average cases
  - No matter what the input array contains, selection sort must always search the unsorted part for its largest remaining element.

# Selection sort: in-place

- Example:

Unsorted part Sorted part

`6 1 4 3 8 7 2 5`

# Selection sort: in-place

- Example:

Unsorted part    Sorted part

`6 1 4 3 5 7 2 8`

# Selection sort: in-place

- Example:

Unsorted part   Sorted part

`6 1 4 3 5 2 7 8`

# Selection sort: in-place

- Example:

Unsorted part Sorted part

`2 1 4 3 5 6 7 8`

# Selection sort: in-place

- Example:

Unsorted part    Sorted part

`1 2 3 4 5 6 7 8`

# Insertion sort.

# Insertion sort

- Like selection sort, **insertion sort** maintains a "sorted part" *S* and "unsorted part" *U* of the input array:

Sorted part Unsorted part

- Insertion sort operates by repeatedly removing the *leftmost* element of *U* and *inserting* it into its "proper place" in *S*.

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
|  | 6 1 4 3 8 7 2 5 |

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# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
| | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
|  | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
| | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |
| 1 6 | 4 3 8 7 2 5 |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
| | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |
| 1 6 | 4 3 8 7 2 5 |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
| | `6 1 4 3 8 7 2 5` |
| `6` | `1 4 3 8 7 2 5` |
| `1 6` | `4 3 8 7 2 5` |
| `1 4 6` | `3 8 7 2 5` |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
| | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |
| 1 6 | 4 3 8 7 2 5 |
| 1 4 6 | 3 8 7 2 5 |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
|  | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |
| 1 6 | 4 3 8 7 2 5 |
| 1 4 6 | 3 8 7 2 5 |
| 1 3 4 6 | 8 7 2 5 |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
| | `6 1 4 3 8 7 2 5` |
| `6` | `1 4 3 8 7 2 5` |
| `1 6` | `4 3 8 7 2 5` |
| `1 4 6` | `3 8 7 2 5` |
| `1 3 4 6` | `8 7 2 5` |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
|  | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |
| 1 6 | 4 3 8 7 2 5 |
| 1 4 6 | 3 8 7 2 5 |
| 1 3 4 6 | 8 7 2 5 |
| 1 3 4 6 8 | 7 2 5 |

# Insertion sort

- Example:

| Sorted part | Unsorted part |
|---|---|
| | 6 1 4 3 8 7 2 5 |
| 6 | 1 4 3 8 7 2 5 |
| 1 6 | 4 3 8 7 2 5 |
| 1 4 6 | 3 8 7 2 5 |
| 1 3 4 6 | 8 7 2 5 |
| 1 3 4 6 8 | 7 2 5 |
| 1 3 4 6 7 8 | 2 5 |
| 1 2 3 4 6 7 8 | 5 |
| 1 2 3 4 5 6 7 8 | Done. |

# Insertion sort

- With insertion sort, most of the “effort” is in *inserting* the element into its proper slot.
  - In contrast, with *selection sort*, most of the effort is in *finding* the largest element to insert.
- Like selection sort, insertion sort too can operate *in-place*:
  - When we remove the leftmost element *x* from *U*, we save it in a temporary variable.
  - To find *x*’s proper “slot”: we “slide down” each element *y* of *S* to the right as long as $y > x$.
  - We finally insert *x*, and repeat until *U* is empty.

# Insertion sort

- Example:

Sorted part Unsorted part

`6 1 4 3 8 7 2 5`

*x*: 6

# Insertion sort

- Example:

Sorted part Unsorted part

`6 1 4 3 8 7 2 5`

# Insertion sort

- Example:

Sorted part Unsorted part

`6 1 4 3 8 7 2 5`

*x*: 1

# Insertion sort

- Example:

Sorted part Unsorted part

6 4 3 8 7 2 5 $x$: 1

Move $y$=6 to the
right because $y$>$x$.

# Insertion sort

- Example:

Sorted part Unsorted part

6 4 3 8 7 2 5 *x*: 1

# Insertion sort

- Example:

Sorted part Unsorted part

`1 6 4 3 8 7 2 5`

# Insertion sort

- Example:

Sorted part Unsorted part

`1 6 4 3 8 7 2 5`

*x*: 4

# Insertion sort

- Example:

Sorted part Unsorted part

1 6 3 8 7 2 5 *x*: 4

Move *y*=6 to the right because *y*>*x*.

# Insertion sort

- Example:

Sorted part Unsorted part

`1    6 3 8 7 2 5`

*x*: 4

# Insertion sort

- Example:

Sorted part Unsorted part

`1 4 6 3 8 7 2 5`

# Insertion sort

- Example:

Sorted part Unsorted part

`1 4 6 3 8 7 2 5`

*x*: 3

# Insertion sort

- Example:

Sorted part Unsorted part

`1 4 6` `8 7 2 5` *x*: 3

Move *y*=6 to the
right because *y*>*x*.

# Insertion sort

- Example:

Sorted part Unsorted part

`1 4   6 8 7 2 5`          *x*: 3

# Insertion sort

- Example:

Sorted part    Unsorted part

1 4   6 8 7 2 5        *x*: 3

Move *y*=4 to the right because *y*>*x*.

# Insertion sort

- Example:

Sorted part Unsorted part

`1   4 6 8 7 2 5`

$x$: 3

# Insertion sort

- Example:

Sorted part Unsorted part

`1 3 4 6 8 7 2 5`

# Insertion sort

- Example:

Sorted part Unsorted part

`1 3 4 6 8 7 2 5`

# Insertion sort

- Example:

Sorted part Unsorted part

`1 3 4 6 7 8 2 5`

# Insertion sort

- Example:

Sorted part Unsorted part

`1 2 3 4 6 7 8` 5

# Insertion sort

- Example:

Sorted part Unsorted part

`1 2 3 4 5 6 7 8`

# Insertion sort

- Pseudocode:

The reason we need this variable is that "sliding" *y* to the right may overwrite the leftmost element of *U*.

```
While U is not empty:
  Save leftmost element x of U into temporary variable.
  Remove x from U.
  Loop from right to left on element y of S:
    If y > x:
      Slide y to the right by one slot.
      Let y be the element to the left of y
    Else (y ≤ x):
      Insert x to the right of y.
      Break
```

Since the algorithm requires only $O(1)$ additional memory (to store *x*), it is still considered to operate "in-place".

# Stability

- Insertion sort is *stable* as long as we "shift over" an element *y* in S if *y* > *x*.

Sorted part Unsorted part

1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ 7 8

$x$: $3_2$

We don't move $y=3_1$ to the
right because *y not* > *x*.

# Stability

- Insertion sort is *stable* as long as we “shift over” an element $y$ in S if $y > x$.

Sorted part Unsorted part

1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8

# Stability

- Insertion sort is *stable* as long as we “shift over” an element *y* in S if $y > x$.

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$ Stable.

# Stability

- If instead we "shift over" *y* whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\quad 7\ 8$ $x$: $3_2$

We move $y=3_1$ to the right because $y \geq x$.

# Stability

- If instead we “shift over” $y$ whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ $3_2$ 7 8
1 2 $3_1$ 7 8 $x$: $3_2$

# Stability

- If instead we “shift over” $y$ whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$

# Stability

- If instead we “shift over” *y* whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$ Not stable.

# Time cost: worst case

```
While U is not empty:
  Save leftmost element x of U into temporary variable.
  Remove x from U.
  Loop from right to left on element y of S:
    If y > x:
      Slide y to the right by one slot.
      Let y be the element to the left of y
    Else (y ≤ x):
      Insert x to the right of y.
      Break
```

- Outer loop executes *n* times.
- Inner loop has to move all the elements of *S* to the right by one slot before inserting *x*.
  - Since *S* grows in size as outer loop iterates, this results in 1, 2, 3, ..., $n$-1 operations.
- $1 + 2 + 3 + \ldots + n\text{-}1 = n(n\text{-}1)/2 = O(n^2)$.
- Worst case is realized when the input are sorted *backwards*.

# Time cost: best case

```
While U is not empty:
  Save leftmost element x of U into temporary variable.
  Remove x from U.
  Loop from right to left on element y of S:
    If y > x:
      Slide y to the right by one slot.
      Let y be the element to the left of y
    Else (y ≤ x):
      Insert x to the right of y.
      Break
```

- Outer loop executes *n* times.
  - Inner loop only executes once -- *x* is inserted as the rightmost element of *S*.
  - This results in only 1 operation per outer loop iteration.
- 1 + 1 + 1 + ... + 1 = $O(n)$.
- Best case is realized when the input is already sorted.

# Insertion sort versus selection sort

- Selection sort is $O(n^2)$ *regardless* of the input.
- Insertion sort is $O(n^2)$ in the worst case and average case, but $O(n)$ in the best case.
  - If your input array *might* already be sorted (or almost sorted), then insertion sort is much better.

# Heapsort.

# Heapsort

- The *heap* data structure we covered earlier in the course turns out to be useful for sorting.
- A *heap* allows the removal of the *largest element* in $O(\log n)$ time.
- To see how this is useful in sorting, recall how *selection sort* operates:

```
While U is not empty:
  Remove the largest element of U and add it to S.
```

# Heapsort

- Selection sort uses a simple *linear search* through *U* to find the largest element in *O*(*n*) time.
- Using a heap, we can do this in *O*(log *n*) time.
- This results in the following **heapsort** algorithm:

```
Build a heap from the data in U.
While U is not empty:
  Remove largest from U and add it to S.
```

# Heapsort

- Building a heap from $n$ data in $U$ takes time at most $O(n \log n)$. *
- The loop iterates $n$ times.
  - Finding+removing largest takes time $O(\log n)$.
- In total, heapsort takes time $O(n \log n) + n*O(\log n) = O(n \log n)$ in both the *worst case* and *best case*.

* It's actually possible to heapify an array of $n$ elements in $O(n)$ time, but that doesn't affect heapsort's *asymptotic* performance.

# Heapsort

- Example:

Unsorted part

Sorted part

`6 1 4 3 8 7 2 5`

First, convert this into an array-based max-heap.

# Heapsort

- Example:

Unsorted part

Sorted part

```
6 1 4 3 8 7 2 5
8 6 7 5 3 4 2 1
```

# Heapsort

- Example:

Unsorted part

Sorted part

```
6 1 4 3 8 7 2 5
8 6 7 5 3 4 2 1
```

Now, repeatedly call `removeLargest()` and add that element to the sorted part.

# Heapsort

- Example:

| Unsorted part | Sorted part |
|---|---|
| 6 1 4 3 8 7 2 5 | |
| 8 6 7 5 3 4 2 1 | |
| 7 6 4 5 3 1 2 | 8 |

Done.

# Heapsort

- Example:

| Unsorted part | Sorted part |
|---|---|
| `6 1 4 3 8 7 2 5` | |
| `8 6 7 5 3 4 2 1` | |
| `7 6 4 5 3 1 2` | `8` |
| `6 5 4 2 3 1` | `7 8` |

Done.

# Heapsort

- Example:

| Unsorted part | Sorted part |
|---|---|
| 6 1 4 3 8 7 2 5 | |
| 8 6 7 5 3 4 2 1 | |
| 7 6 4 5 3 1 2 | 8 |
| 6 5 4 2 3 1 | 7 8 |
| 5 3 4 2 1 | 6 7 8 |

Done.

# Heapsort

- Example:

| Unsorted part | Sorted part |
|---|---|
| 6 1 4 3 8 7 2 5 | |
| 8 6 7 5 3 4 2 1 | |
| 7 6 4 5 3 1 2 | 8 |
| 6 5 4 2 3 1 | 7 8 |
| 5 3 4 2 1 | 6 7 8 |
| 4 3 1 2 | 5 6 7 8 |

Done.

# Heapsort

- Example:

| Unsorted part | Sorted part |
|---|---|
| 6 1 4 3 8 7 2 5 | |
| 8 6 7 5 3 4 2 1 | |
| 7 6 4 5 3 1 2 | 8 |
| 6 5 4 2 3 1 | 7 8 |
| 5 3 4 2 1 | 6 7 8 |
| 4 3 1 2 | 5 6 7 8 |
| 3 2 1 | 4 5 6 7 8 |
| 2 1 | 3 4 5 6 7 8 |
| 1 | 2 3 4 5 6 7 8 |
| | Done. 1 2 3 4 5 6 7 8 |

# Heapsort

- In the way that Heapsort was demonstrated in the previous slides, it could be implemented as follows:

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>();
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- In the way that Heapsort was demonstrated in the previous slides, it could be implemented as follows:

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>();
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

- This works fine, but does not operate *in-place* because the heap requires *O*(*n*) *additional* storage to the input array itself.

# Heapsort

- In the way that Heapsort was demonstrated in the previous slides, it could be implemented as follows:

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>();
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

```
class HeapImpl12<T ...> ... {
  T[] _nodeArray;
  int _numNodes = 0;

  HeapImpl12<T> () {
    _nodeArray = (T[]) new Comparable[128];
  }
}
```

`_nodeArray` requires $O(n)$ *additional* storage to the user's input `array`

# Heapsort

- However, we can also implement Heapsort to work *in-place*.
- The "trick" is that the *input array* to Heapsort will serve as the **`HeapImpl12`**'s *underlying storage*.
  - In **`HeapImpl12`**, instead of allocating a *new* **`_nodeArray`**, we will simply set it to the the user's input **`array`**.

# Heapsort

- We implement a new **`HeapImpl12`** constructor and use it in **`heapsort`**:

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

```
class HeapImpl12<T ...> ... {
  T[] _nodeArray;
  int _numNodes = 0;

  HeapImpl12<T> () {
    _nodeArray = (T[]) new Comparable[128];
  }
  HeapImpl12<T> (T[] array) {
    _nodeArray = array;
  }
}
```

Here we are offering a second constructor, in which the user can pass in an *existing* array as the heap's underlying storage. ==> No extra storage!

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

  **`6 1 4 3 8 7 2 5`**

- The array above *is* the heap's underlying storage (**`_nodeArray`**), and it is *also* the user's input **`array`**.
- Initially, **`_numNodes`** = 0.
- Each time we "add" an element to the heap, **`_numNodes`** will increase by 1.

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 1 4 3 8 7 2 5**

**_numNodes:** 0

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 1 4 3 8 7 2 5**

`_numNodes: 1`

Increment `_numNodes`

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 1 4 3 8 7 2 5**

`_numNodes: 1`

Call `bubbleUp`

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 1 4 3 8 7 2 5**

**_numNodes:** 2

Increment **_numNodes**

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements     Non-heapified elements

**6 1 4 3 8 7 2 5**

`_numNodes: 2`

Call **`bubbleUp`**

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements    Non-heapified elements

**6 1 4 3 8 7 2 5**

**_numNodes:** 3

Increment **_numNodes**

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 1 4 3 8 7 2 5**

`_numNodes: 3`

Call `bubbleUp`

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let’s turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 1 4 3 8 7 2 5**

`_numNodes: 4`

Increment `_numNodes`

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 1 4 3 8 7 2 5**

**_numNodes: 4**

Call **bubbleUp**

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 3 4 1 8 7 2 5**

**_numNodes: 4**

Call **bubbleUp**

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 3 4 1 8 7 2 5**

**_numNodes:** 5

Increment **_numNodes**

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**6 3 4 1 8 7 2 5**

`_numNodes: 5`

Call `bubbleUp`

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**8 6 4 1 3 7 2 5**

**_numNodes: 5**

Call **bubbleUp**

Keep repeating this process...

```
void heapsort (Integer[] array) {
  Heap12<Integer> heap = new HeapImpl12<Integer>(array);
  for (int i = 0; i < array.length; i++) {
    heap.add(array[i]);
  }
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

**`8 6 7 5 3 4 2 1`**

**`_numNodes: 8`** Done.

- We have now constructed a heap *within the input array itself.*
  - This requires 0 extra storage.

# Heapsort

- However, we're still not done.
- We still have to call **`removeLargest()`** repeatedly, and store its result into the *leftmost* position of the *sorted part* of the array.
- Since we're operating *in-place*, this will require that store the largest value *x* in a temporary variable.

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

**8 6 7 5 3 4 2 1**

**`_max:`**

**`_numNodes: 8`**

Save the heap's largest element in **`_max`**, and then remove the largest element.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

**8 6 7 5 3 4 2 1**

**`_max`**: **8**

**`_numNodes`**: **8**

Save the heap's largest element in **`_max`**, and then remove the largest element.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

**1 6 7 5 3 4 2**

Calling `removeLargest` requires us to `trickleDown` from the root.

**_max: 8**

**_numNodes: 7**

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part    Sorted part

**7 6 1 5 3 4 2**

**_max: 8**
**_numNodes: 7**

Calling `removeLargest` requires us to `trickleDown` from the root.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part Sorted part

**7 6 4 5 3 1 2**

**_max: 8**
**_numNodes: 7**

Calling `removeLargest` requires us to `trickleDown` from the root.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

**7 6 4 5 3 1 2 8**

**_max:**
**_numNodes: 7**

Finally, we store **`_max`** into the *sorted part* of the array.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

**7 6 4 5 3 1 2 8**

**_max:** 7

**_numNodes:** 7

Save the heap's largest element in **`_max`**, and then remove the largest element.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part　　　Sorted part

**2 6 4 5 3 1**　　**8**

**_max: 7**

**_numNodes: 7**

Calling `removeLargest` requires us to `trickleDown` from the root.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part Sorted part

**6 2 4 5 3 1 8**

Calling `removeLargest` requires us to `trickleDown` from the root.

**_max: 7**
**_numNodes: 7**

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part        Sorted part

**6 5 4 2 3 1    8**

**`_max: 7`**
**`_numNodes: 7`**

Calling **`removeLargest`** requires us to **`trickleDown`** from the root.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

**6 5 4 2 3 1 7 8**

**`_max:`**
**`_numNodes: 7`**

Finally, we store **`_max`** into the *sorted part* of the array.

```
void heapsort (Integer[] array) {
  ...
  for (int i = array.length - 1; array >= 0; array--) {
    array[i] = heap.removeLargest();
  }
}
```

# Heapsort

- We repeatedly remove the largest element and store it into the leftmost slot of the unsorted part of the array, *until the heap is empty*.
- At that point, the array will be completely sorted.
- Since this required only one auxiliary variable (**`_max`**), the algorithm works *in-place*.

# Heapsort

- In summary, heapsort is an in-place sorting algorithm whose best and worst case time costs are $O(n \log n)$.
- However, the algorithm is *not* stable because the heap ordering may cause the relative order of duplicate elements to become inverted.

# Mergesort.

# Approach 2: divide and conquer

- So far we've looked at sorting algorithms that partition the input array into a *sorted part* and *unsorted part*, and then "grow" the sorted part to be the entire array.
- An alternative approach altogether is based on the *divide-and-conquer* principle:
  - To sort a list of size *n*:
    - Divide the list into two halves (approx. size *n*/2).
    - Sort each half independently using recursion.
    - Combine the 2 sorted lists of *n*/2 elements into 1 sorted list of *n* elements.

# Mergesort

- The first algorithm we examine that uses divide-and-conquer is **Mergesort**.
- Here's the "main idea" behind the algorithm:
  - Suppose we have a *left list* and a *right list* that are *already sorted*.
  - To combine these two lists into one larger sorted list, we just:
    - Iterate through both lists simultaneously.
    - "Pick out" the smaller element from the current position of either the left or right list, and insert it into our *combined* list.

# Merging two sorted lists

- Example:

Left list Right list Combined list

`1 3 4 6 2 5 7 8`

`i` `j` `k`

Iterate through both lists:

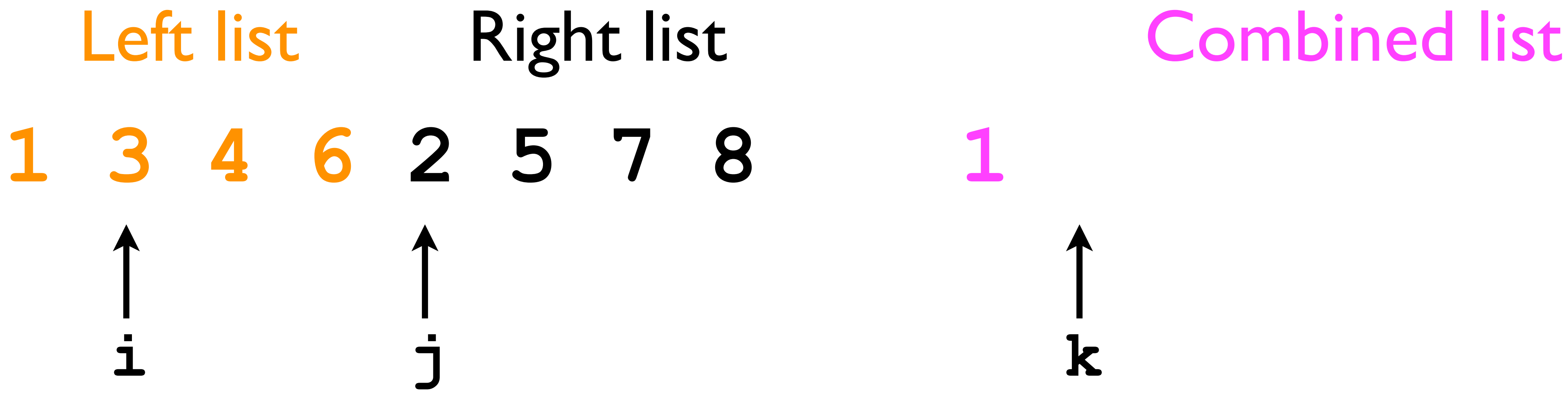
*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8 1`

`i j k`

Iterate through both lists:

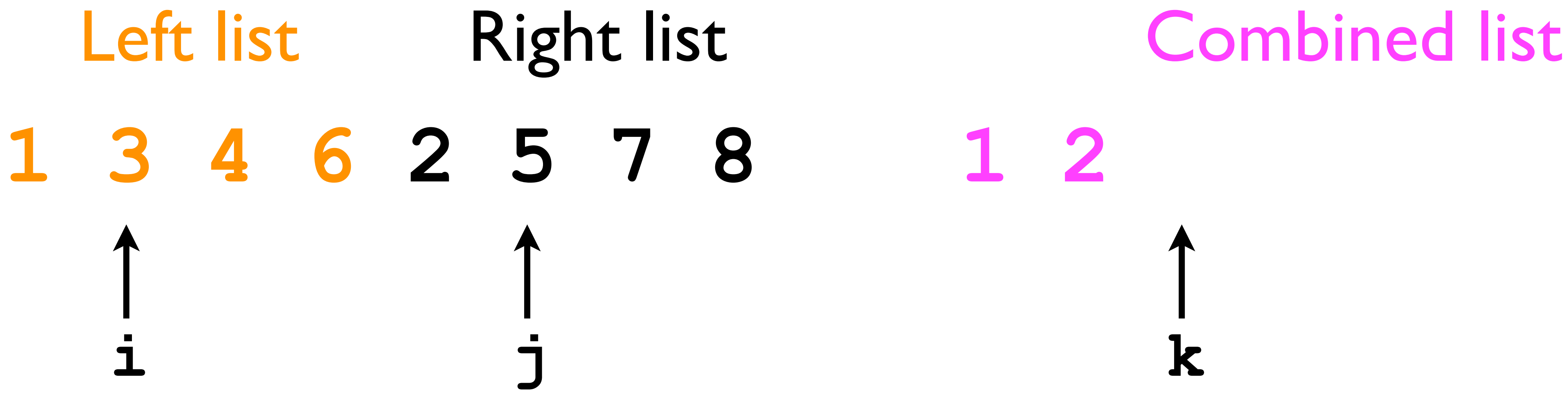
*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:



Iterate through both lists:

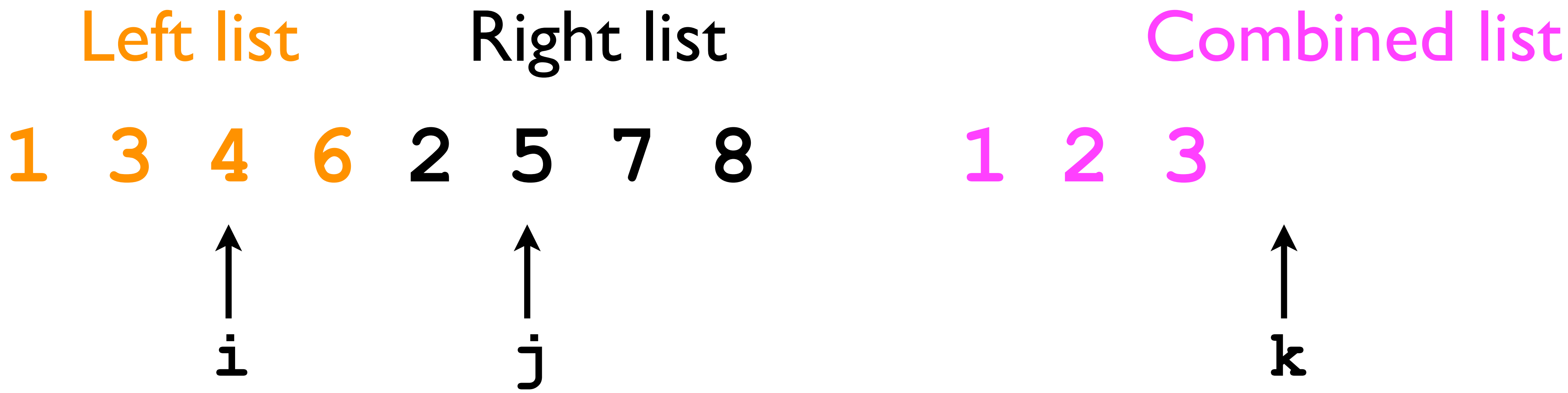
*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8`  `1 2 3`

`i` `j` `k`

Iterate through both lists:

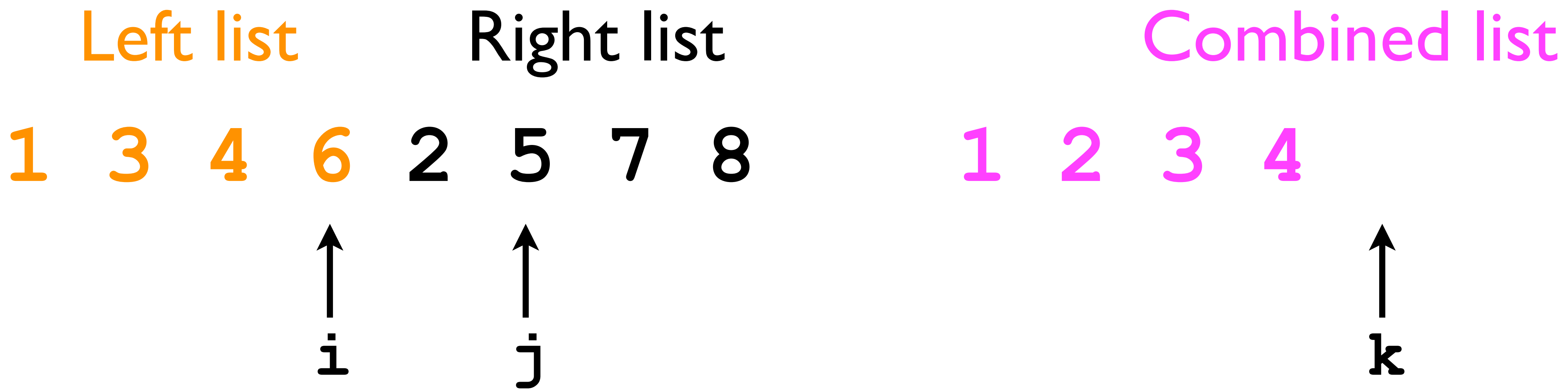
*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:



Left list Right list Combined list

**`1 3 4 6 2 5 7 8`** **`1 2 3 4`**

**`i`** **`j`** **`k`**

Iterate through both lists:

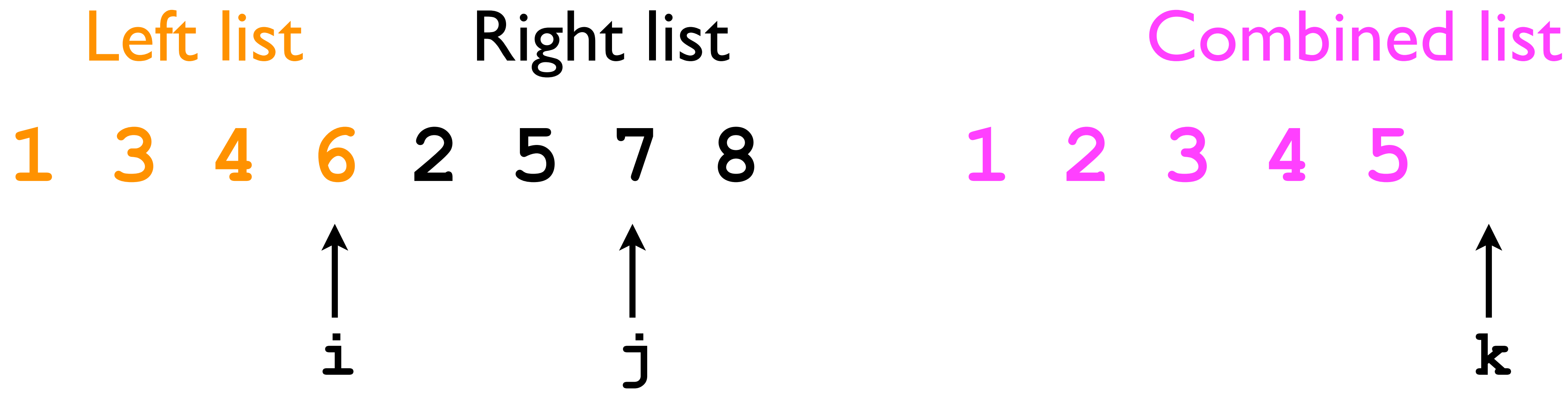
*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8` `1 2 3 4 5`

`i` `j` `k`

Iterate through both lists:

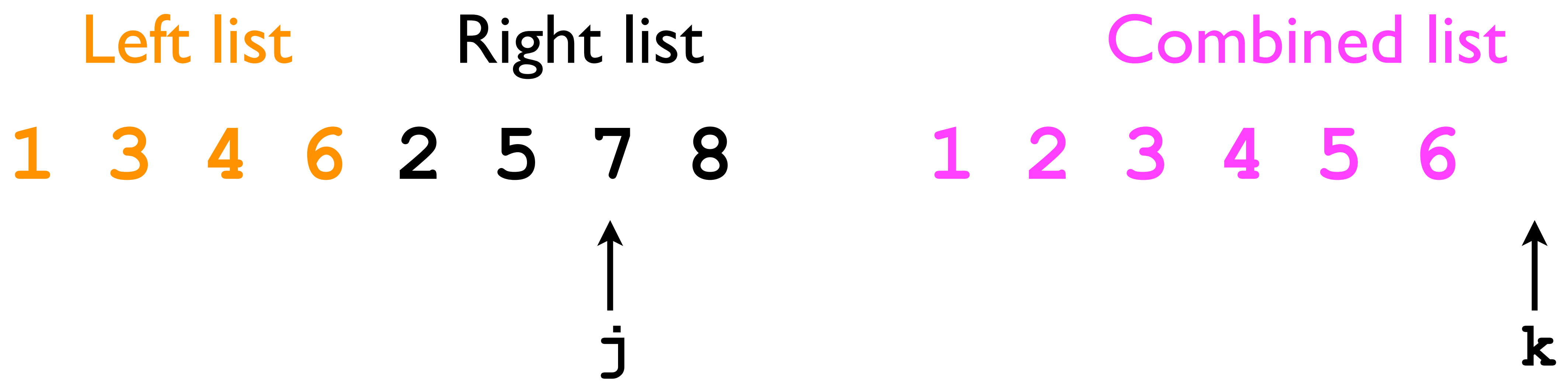
*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:



Left list Right list

Combined list

1 3 4 6 2 5 7 8

1 2 3 4 5 6

j

k

Iterate through both lists:

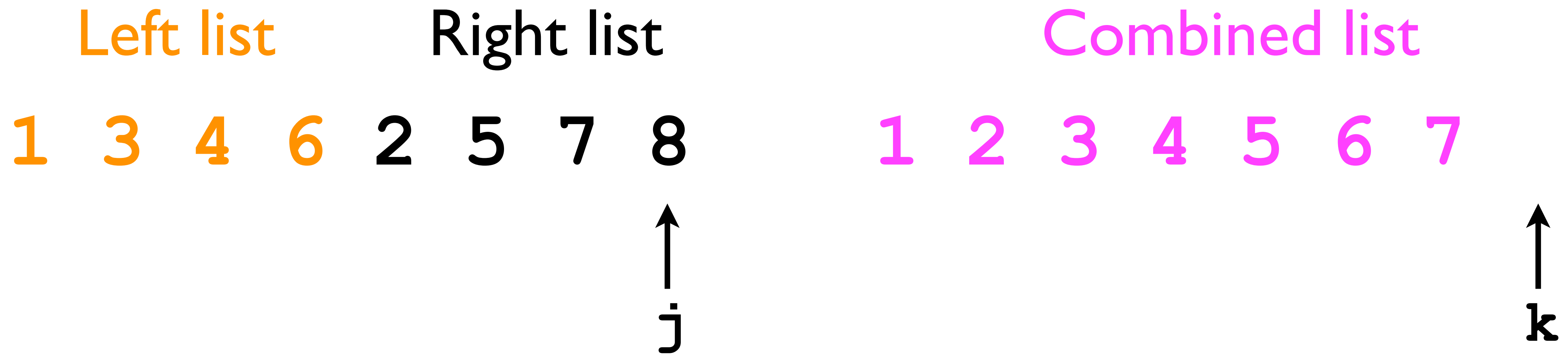
*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:



Left list Right list Combined list

1 3 4 6 2 5 7 8 1 2 3 4 5 6 7

j k

Iterate through both lists:

*Pick out* the smaller element *x* from the current position of either the left or right list;

*Advance* the pointer of whichever list contained x;

Then *insert x* into the combined list.

# Merging two sorted lists

- Example:

Left list Right list Combined list

`1 3 4 6 2 5 7 8` `1 2 3 4 5 6 7 8`

Done.

Iterate through both lists:
*Pick out* the smaller element *x* from the current position of either the left or right list;
*Advance* the pointer of whichever list contained x;
Then *insert x* into the combined list.

# Mergesort

- Given a left list (*n*/2 elements) and a right list (*n*/2 elements), "merging" them into a combined list (*n* elements) takes time O(*n*).
- However, it requires that we allocate a *temporary array* of size *n*.
  - Mergesort does not operate *in-place*.*
- After merging, we copy the elements in the temporary array back into the input array.

* Except when using a linked-list representation.

# Mergesort

- Given a procedure to *merge two sorted lists*, we can define a *recursive sorting algorithm* in the following way:
  - Given an input array:
    - If its length is 1, then it’s already sorted.
    - Else:
      - Divide the list into two halves.
      - Recursively sort each half.
      - Merge their results into one combined list.

# Mergesort

- Mergesort's pseudocode:

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Base case

Recursive part

- Let's see how it works in practice...

# Mergesort

- Example: First stage: recursively divide until we reach the base case.

**6 1 4 3 8 7 2 5**

Split list and recurse.

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example: First stage: recursively divide until we reach the base case.

**6 1 4 3 8 7 2 5**

**6 1 4 3** **8 7 2 5**

Split list and recurse.

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example: First stage: recursively divide until we reach the base case.

6 1 4 3 8 7 2 5 — Split list and recurse.

6 1 4 3    8 7 2 5 — Split list and recurse.

6 1    4 3    8 7    2 5 — Split list and recurse.

6    1    4    3    8    7    2    5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example:

Each of these is a "list" (size 1) passed to a recursive call to Mergesort.

**6 1 4 3 8 7 2 5**

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example:

Second stage: merge each pair of sorted sub-lists.

1 6    3 4    7 8    2 5

Merge the two sub-lists.

6 1 4 3 8 7 2 5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example:

Second stage: merge each pair of sorted sub-lists.

1 3 4 6        2 5 7 8

Merge the two sub-lists.

1 6    3 4    7 8    2 5

6  1  4  3  8  7  2  5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example:

Second stage: merge each pair of sorted sub-lists.

1 2 3 4 5 6 7 8

Merge the two sub-lists.

1 3 4 6        2 5 7 8

1 6    3 4    7 8    2 5

6    1    4    3    8    7    2    5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example:

Done.

```
            1 2 3 4 5 6 7 8

        1 3 4 6         2 5 7 8

      1 6     3 4     7 8     2 5

    6   1   4   3   8   7   2   5
```

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

# Mergesort

- Example:

```
       1 2 3 4 5 6 7 8

   1 3 4 6         2 5 7 8

 1 6     3 4     7 8     2 5

6   1   4   3   8   7   2   5
```

- The *depth* of this recursive call stack is the *number of times we can divide* n *by 2*, i.e., *O*(log *n*).

# Mergesort

- Example:

**1 2 3 4 5 6 7 8**

**1 3 4 6** **2 5 7 8**

**1 6** **3 4** **7 8** **2 5**

**6** **1** **4** **3** **8** **7** **2** **5**

- At each level, *each element* in the input array had to be "touched" *once* (for the `merge` operation).
- *In total*: $O(\log n) * n = O(n \log n)$.

# Mergesort

- Because Mergesort's dividing and merging requires the same number of operations *regardless* of the particular input, Mergesort's *best case* and *worst case* time complexities are both *O*(*n* log *n*).
- Mergesort is *stable* as long as the `merge` procedure selects the left array's *x* in the case of ties.