CSE 12:
Basic data structures and object-oriented design

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Lecture Twelve
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More on generics.

Collections to hold data of type `T`

- Up to now we have discussed generics in its simplest usage -- store data of an arbitrary type `T` in a container.
    - This worked fine for lists/arrays/stacks/queues, in which we ignore any *order relations* among the elements.
- Sometimes, however, the type `T` cannot be "just any old `Object`" -- type `T` must sometimes *satisfy some conditions*.

Constraints on `T`

- An example of this is the `HeapImpl12` class you are building for P4.
    - The elements must all be `Comparable` -- the heap implementation needs to be able to call `compareTo(o)` on every element stored in the tree.
    - If we place no restrictions on `T`, then the Java compiler cannot guarantee that an arbitrary element of the `_nodeArray` will actually be `Comparable`.

Constraints on `T`

- Suppose we add three objects to a heap:

```
heap = new Heap12<Object>();
heap.add(“Michael”);  // OK: String is Comparable
heap.add(“Bolton”);  // OK: String is Comparable
heap.add(new Object()); // Not OK: Object not Comparable
```

- Internally, the **`HeapImpl12`** class will need to call **`compareTo`** on all objects to implement **`bubbleUp`** and **`trickleDown`**, e.g.:

```
if (_nodeArray[idx1].compareTo(_nodeArray[idx2]) < 0) {
  ...
}
```

But if `idx1` refers to the `Object` we added, this method will fail because `Object` does not implement the `Comparable` interface.

Bounds on type parameters

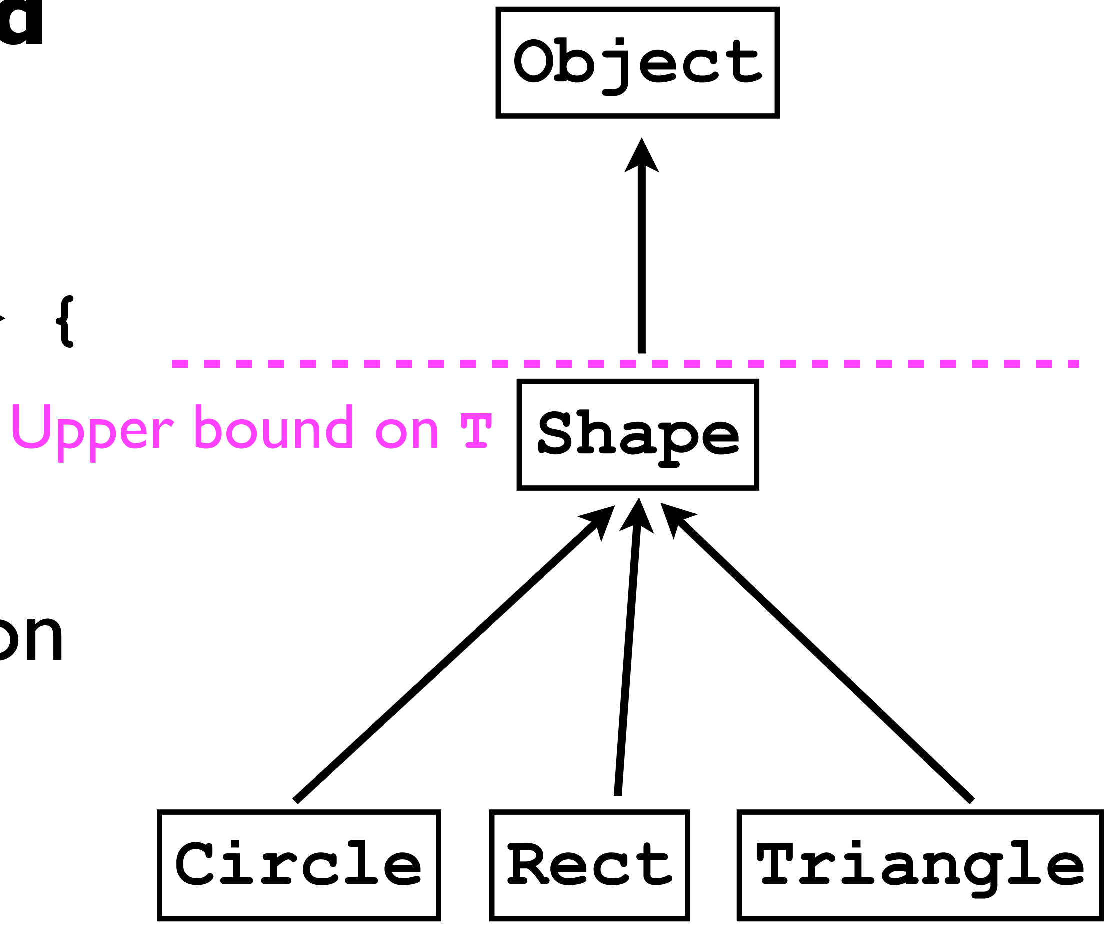
- What we want is a way of *enforcing* that the type parameter **`T`** allowed by the **`HeapImpl12`** class -- as well as the **`Heap12`** interface itself -- be of type **`Comparable`**.
- Java generics facilitates these constraints on **`T`** by supporting **bounds** on type parameters.
- Suppose, when implementing a generic class with type parameter **`T`**, we want to *ensure* that **`T`** must be *some subclass* of a class **`A`**.
    - *Example*: we want to implement a container for **`Shape`** objects -- we don't care what *particular* kind of **`Shapes`** they are, so long as they all *inherit from* the **`Shape`** class.

Bounds on type parameters

- To implement a generic class with the guarantee that type parameter `T` is a `Shape`, we can use an **upper bound** on `T`:

```
class MyContainer<T extends Shape> {
   ...
}
```

- Here, `Shape` is the **upper bound** on type parameter `T`.
  - `MyContainer` can only be instantiated when `T` is `Shape`, or *any sub-class of* `Shape`.



`Object`

Upper bound on `T`

`Shape`

`Circle`

`Rect`

`Triangle`

Bounds on type parameters

- Given this upper bound on `T`, the Java compiler will enforce that `T` be of type `Shape`:

```
MyContainer<Shape> container1 =
  new MyContainer<Shape>();  // OK

MyContainer<Circle> container2 =
  new MyContainer<Circle>();  // OK

MyContainer<Object> container4 =
  new MyContainer<Object>();  // Not OK

          Compiler error message:
            type parameter java.lang.Object is not within its bound
            MyContainer<Object> container4 = new MyContainer<Object>();

MyContainer<Student> container3 =
  new MyContainer<Student>();  // Not OK
```

Bounds on type parameters

- We can also require that type **`T`** *implement* some interface.
    - For example, a **`HeapImpl12`** should only store elements that are all **`Comparable`**.
- Java generics gives us this power:

```
class HeapImpl12<T extends Comparable> implements Heap12<T> {
   ...
}
```

- The "**`extends Comparable`**" enforces that any **`T`** we pass in as the type parameter *must* be of type **`Comparable`**.
    - Since **`Comparable`** is an *interface*, this means that type **`T`** must *implement* the interface **`Comparable`** (even though we use the word "**`extends`**").

Bounds on type parameters

- With this restriction on **`T`** in place, we can no longer instantiate a **`HeapImpl12`** with a type parameter **`T`** that does not implement **`Comparable`**:

```
// String and Integer are both Comparable
HeapImpl12<String> heap1 = new HeapImpl12<String>();  // OK
HeapImpl12<Integer> heap2 = new HeapImpl12<Integer>(); // OK

// Next line won't compile because Object is not Comparable
HeapImpl12<Object> heap3 = new HeapImpl12<Object>();
```

- The Java *compiler* will prevent us from instantiating a heap with a non-**`Comparable`** type.

- We may also wish to define the *interface* **`Heap12`** to accept only those types **`T`** that implement **`Comparable`**:

```
interface Heap12<T extends Comparable> {
  ...
}
```

Bounds on type parameters

- In the previous example, `Comparable` was the upper bound of `T`.
- The `Comparable` interface takes a type parameter of its own.

```
interface Comparable<U> {
  int compareTo (U o);
}
```

(In the previous example, we used the `Comparable` interface in “compatibility mode”, where we did not specify `U`).

- The type parameter `U` specifies what kinds of objects `o` we should be able to compare to.

Bounds on type parameters

- By offering **bounds** on type parameters, Java also gives us the power to define what kinds of objects `U` we can `compareTo`, *in terms of the type* `T` *we've already defined.*
- Example:

```
class HeapImpl12<T extends Comparable<T>> ... {
  ...
}
```

- Here, we require that whatever type `T` the `HeapImpl12` is instantiated with, it *must* be `Comparable` to *other objects of type* `T`.

Bounds on type parameters

- Consider the following example:

```
class B { }
class A implements Comparable<B> {
  int compareTo (B o) {
    return 0;
  }
}
```

- Given the definitions above, an object of type A can *only* be compared to objects of type **`B`**.

```
final A a = new A();
final B b = new B();
final int result = a.compareTo(b);  // OK
```

  - We *cannot* compare **`a`** to another object of type **`A`**!

Bounds on type parameters

- Given our definition of HeapImpl12,

  ```
  class HeapImpl12<T extends Comparable<T>> ... {
     ..
  }
  ```

  if we try to instantiate a **`HeapImpl12`** with **`A`** as the type parameter...

  ```
  HeapImpl12<A> heap = new HeapImpl12<A>();
  ```

  ... the compiler will complain:

  ```
    type parameter A is not within its bound
     HeapImpl12<A> h = new HeapImpl12<A>();
  ```

- This error occurs because, even though **`A`** is **`Comparable`** to *something* (**`B`**), it is not **`Comparable<A>`**.

Bounds on type parameters

- On the other hand,
  - `String` implements `Comparable<String>`
  - `Integer` implements `Comparable<Integer>`
- Both `String` and `Integer` would be accepted as type parameters for `HeapImpl12`:

```
HeapImpl12<String> h1 = new HeapImpl12<String>();
HeapImpl12<Integer> h2 = new HeapImpl12<Integer>();
```

Both are OK

Bounds on type parameters

- While useful, our current definition of **`HeapImpl12`** is a bit *overly restrictive*.
- Consider a hierarchy of **`Shape`** classes:

```
class Shape implements Comparable<Shape> {
   int compareTo (Shape o) { ... }
}
class Rectangle extends Shape {
  ...
}
```

- The **`Rectangle`** class inherits the **`compareTo (Shape o)`** method from its parent **`Shape`** class.

Bounds on type parameters

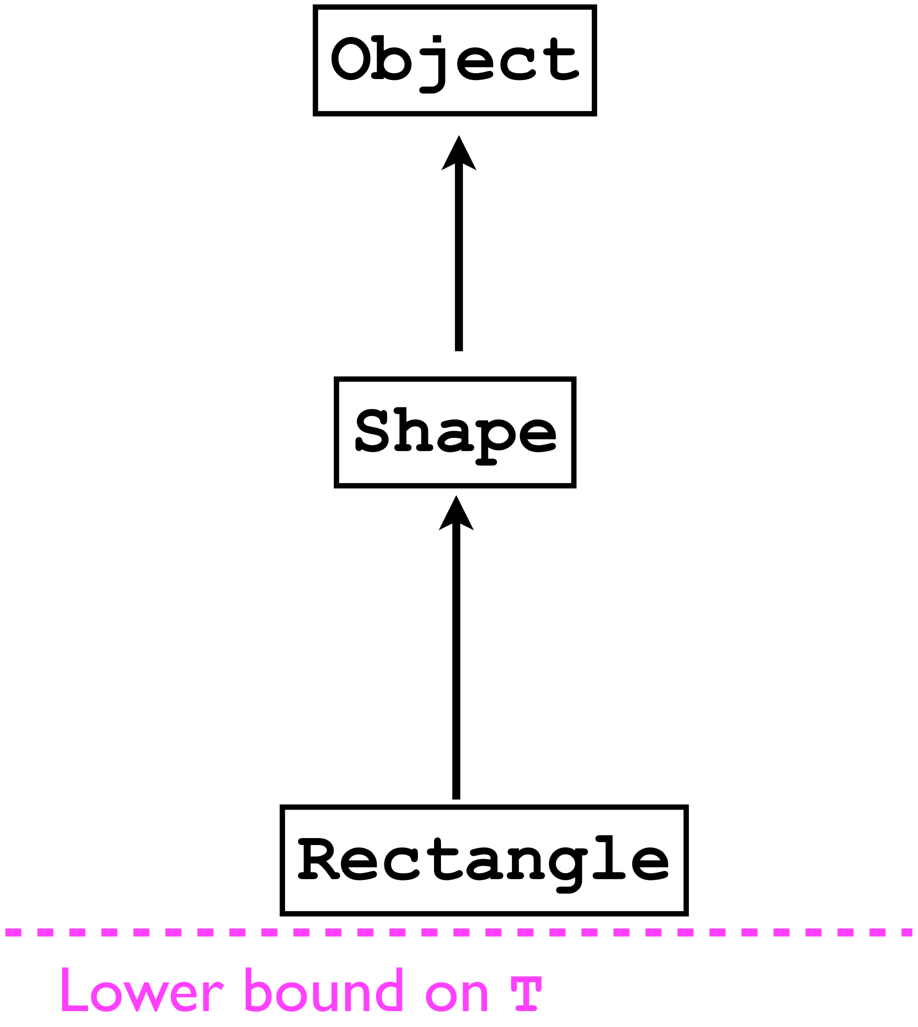
- However, **`Rectangle`** does not offer a method **`compareTo (Rectangle o)`** designed specifically for other **`Rectangle`** objects.
- Hence, the **`Rectangle`** class *could not be used* as the type parameter **`T`** when instantiating a **`HeapImpl12`**:

```
class HeapImpl12<T extends Comparable<T>> ...
```

  - *Reason*: Even though **`Rectangle`** is **`Comparable`** to other **`Shape`** objects, it is not **`Comparable<Rectangle>`**.
    - I.e., **`Rectangle`** offers no **`int compareTo (Rectangle o)`** method.

Lower bounds on types

- What we need is a way of expressing that type parameter `T` may be `Comparable` with class `T`, *or any super-class of* `T`.
  - E.g., we want to allow `HeapImpl12` to store `Rectangle` objects:
    - `Rectangle`s are all `Comparable` with `Shape`, where `Shape` is a *super-class* of `Rectangle`.
- To solve this problem, Java offers **lower bounds** on type parameters.



Lower bounds on types

- For example, we can allow the **`HeapImpl12`** class to accept any type **`T`** so long as **`T`** is **`Comparable`** to class **`T`**, or any super-class of **`T`**.

```
class HeapImpl12<T extends Comparable<? super T>> ... {
   ...
}
```

- The wildcard type **`?`** indicates:
  - "We don't care which type **`T`** is **`Comparable`** to, so long as it's **`Comparable`** to some super-class of **`T`** (or **`T`** itself)."
    - The keyword **`super`** indicates the **lower bound** of the type parameter.

Lower bounds on types

- Given this revised definition of `HeapImpl12`, we can now instantiate a heap of `Rectangle` objects:

```
HeapImpl12<Rectangle> heap =
  new HeapImpl12<Rectangle>();  // OK
```

Binary search trees

Still something to be desired

- Heaps offer fast access to the largest element in a collection.
  - This is most useful in a priority queue.
- However, finding an *arbitrary* element is still slow -- $O(n)$ time.
- We may want to sacrifice efficiency of access to the *largest* access in exchange for increased efficiency to access any *arbitrary* element.

Binary search trees

- A **binary search tree** (BST) is a binary-tree based data structure that offers *O*(log *n*) *average*-case time costs for:

  ```
  add(o)
  find(o)
  remove(o)
  findLargest/removeLargest(o)
  ```

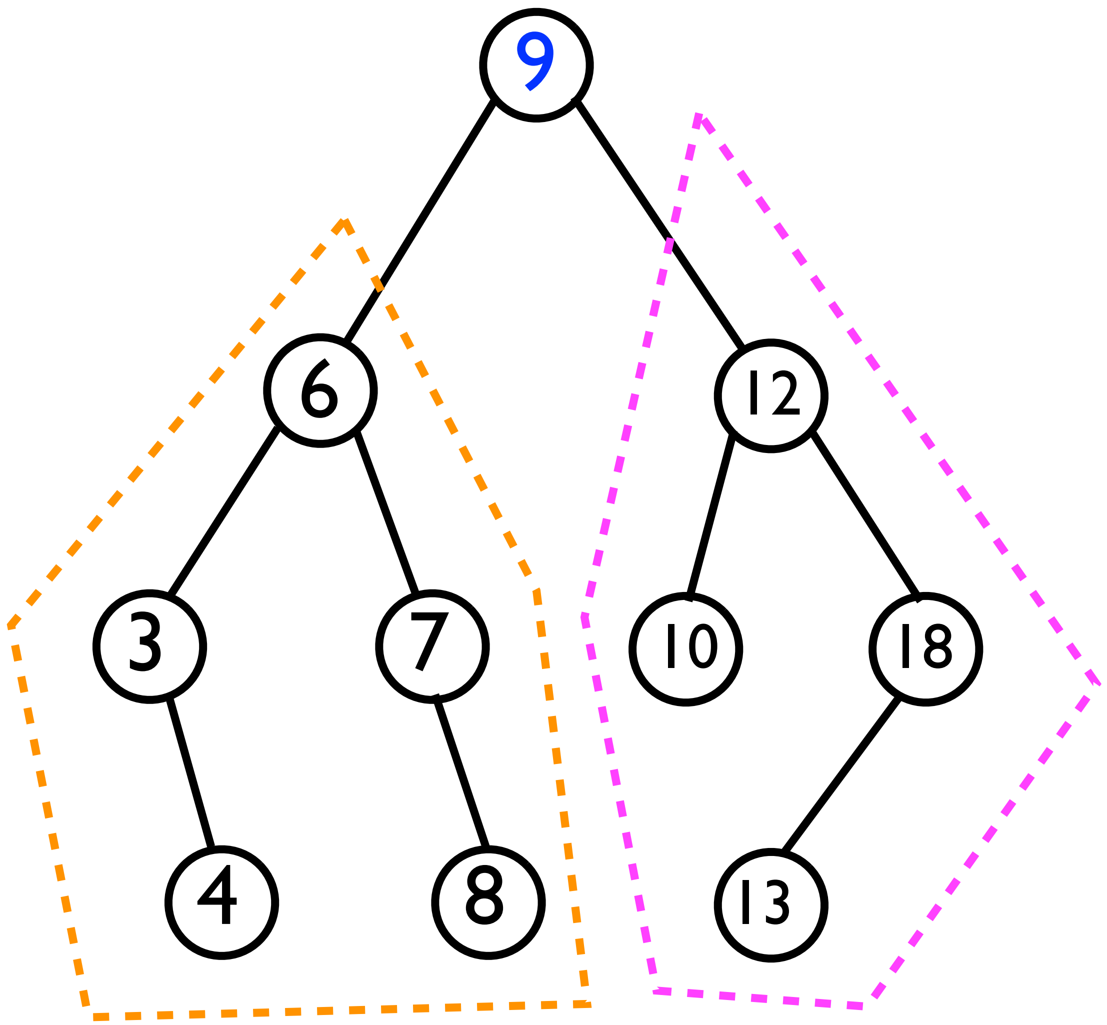
- As with heaps, BSTs exploit the order relations among elements.
  - Heaps required the *root* node r of each sub-tree to be no smaller than any *descendant* node of *r*.
  - BSTs impose constraints on the magnitude of nodes in the *left sub-tree* compared to the magnitude of nodes in the *right sub-tree*.

Binary search trees

- More specifically, a binary search tree (BST) is a binary tree (not necessarily complete) that has the following (recursive) *ordering property*:
    - For each node *n*:
        - All nodes in the *left sub-tree* of *n* are "less than" node *n* itself.
        - All nodes in the *right sub-tree* of *n* are "greater than or equal to" node *n* itself.
        - Both the left and right sub-trees are themselves BSTs.

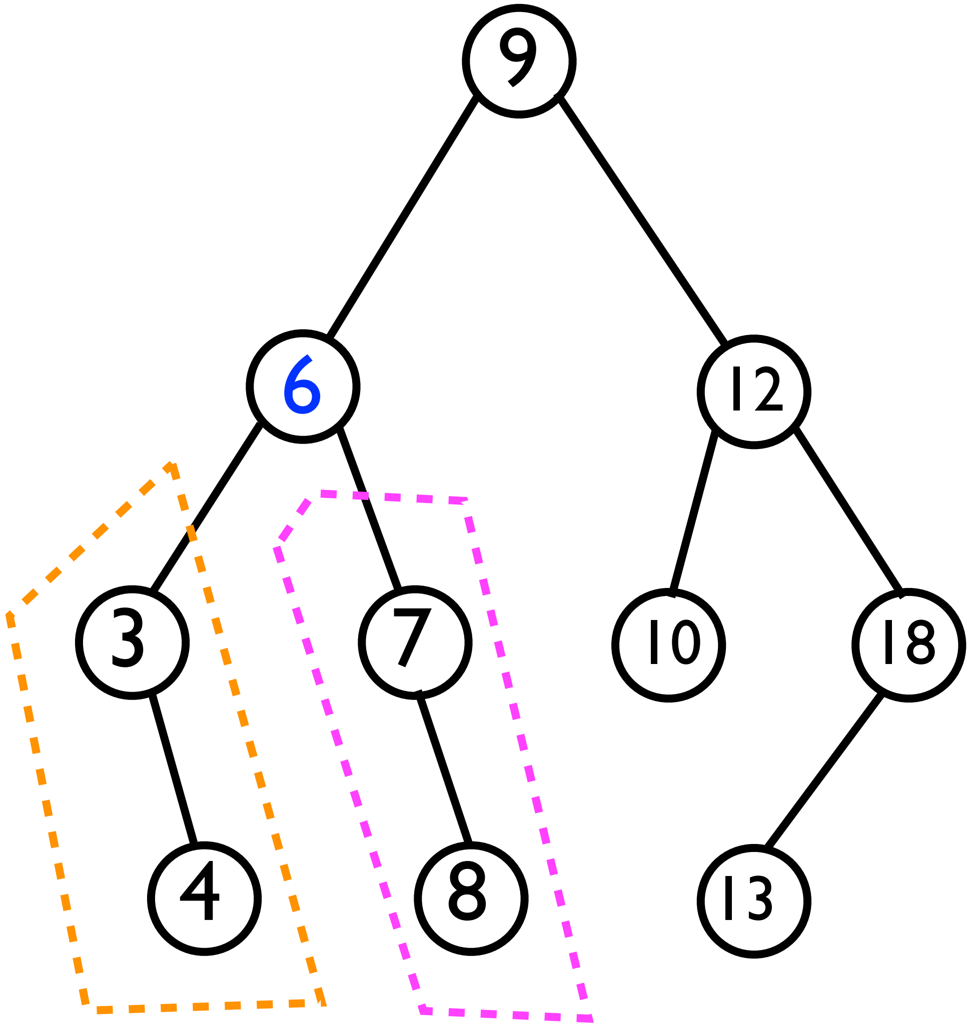
Binary search trees

Left sub-tree < Node (9) ≤ Right sub-tree

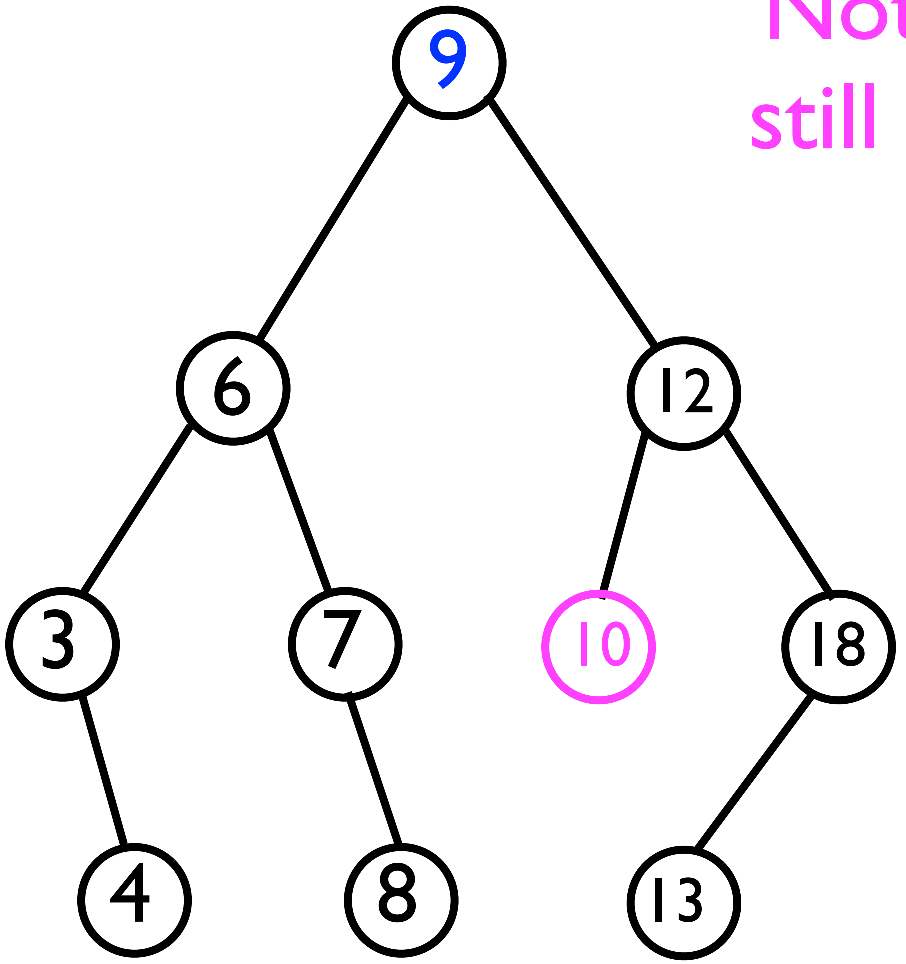


Binary search trees

Left sub-tree < Node (6) ≤ Right sub-tree



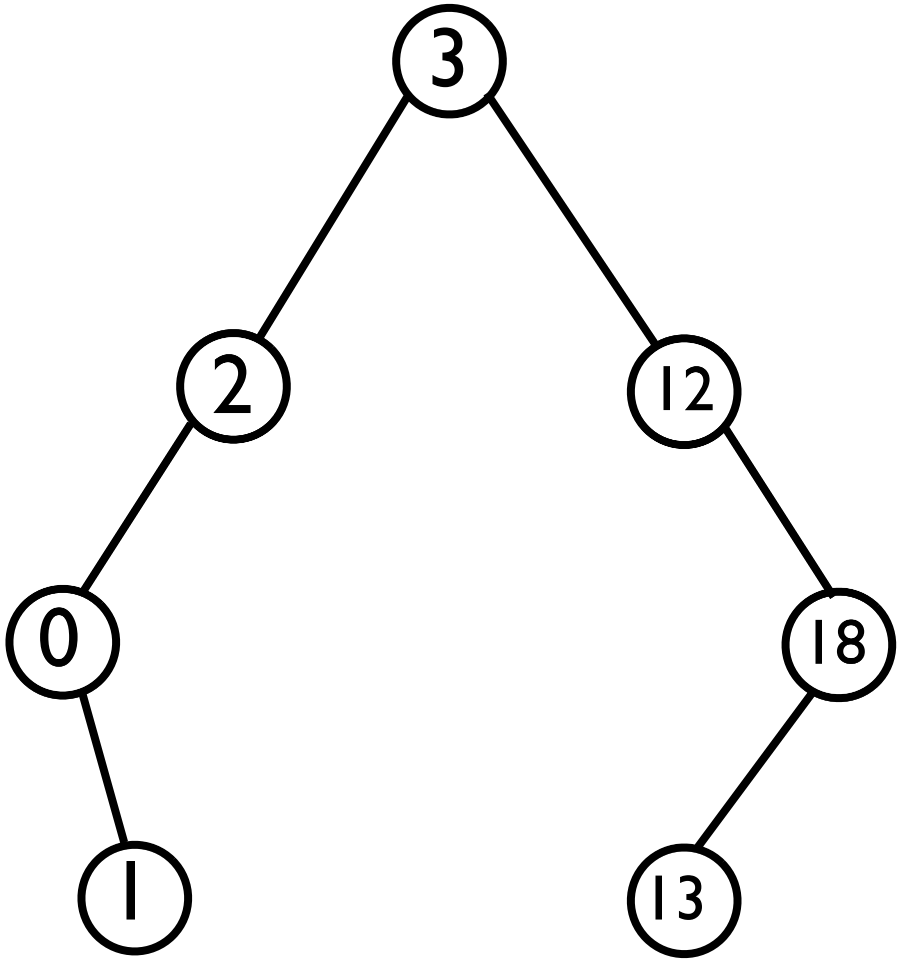
Binary search trees



Note that this node must still be greater or equal to 9!

Binary search trees

Which of these trees are valid BSTs?



Binary search trees

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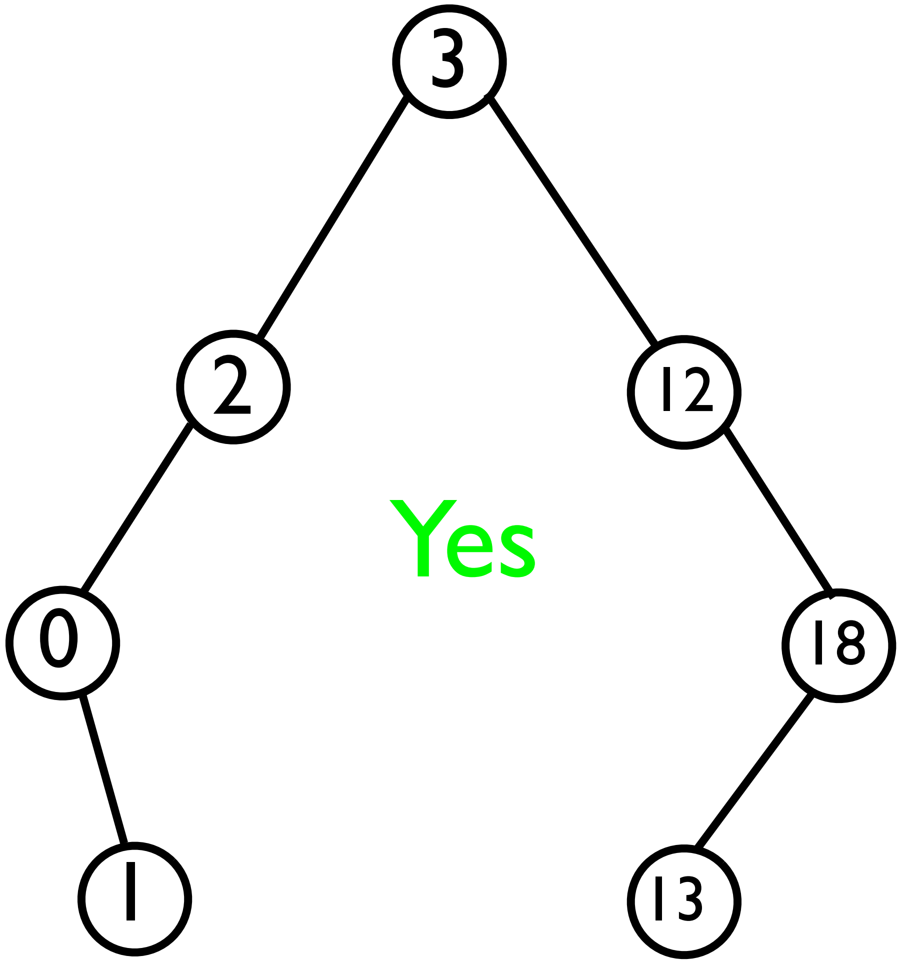
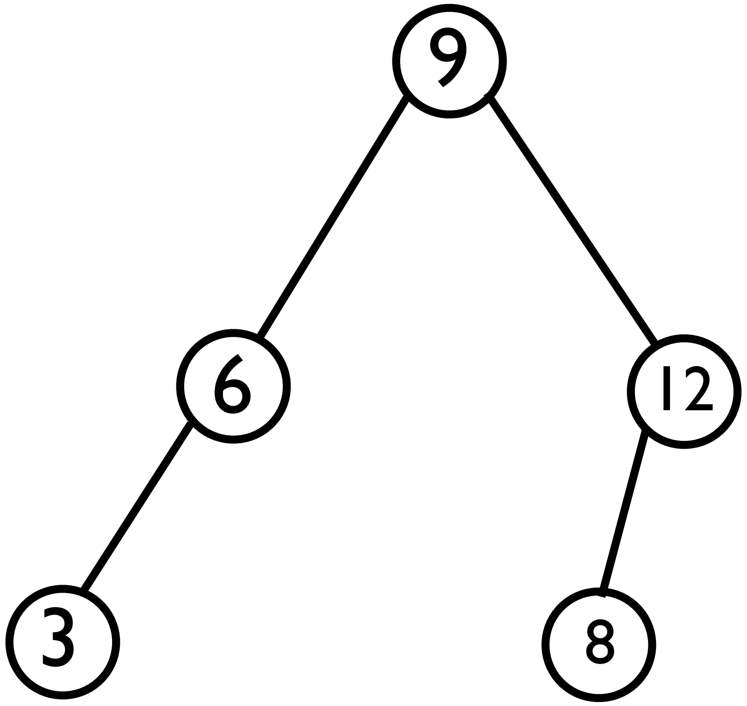


Yes

Binary search trees

Which of these trees are valid BSTs?



Binary search trees

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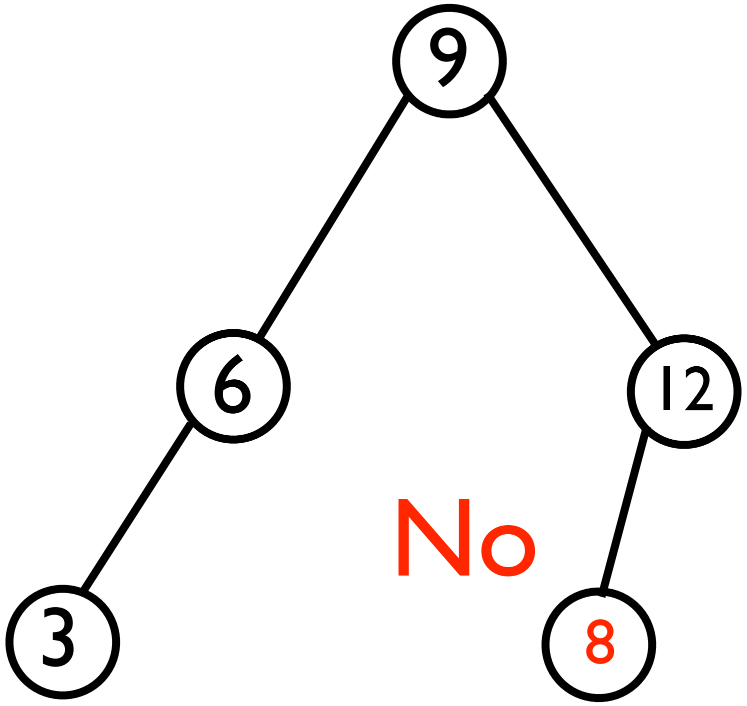
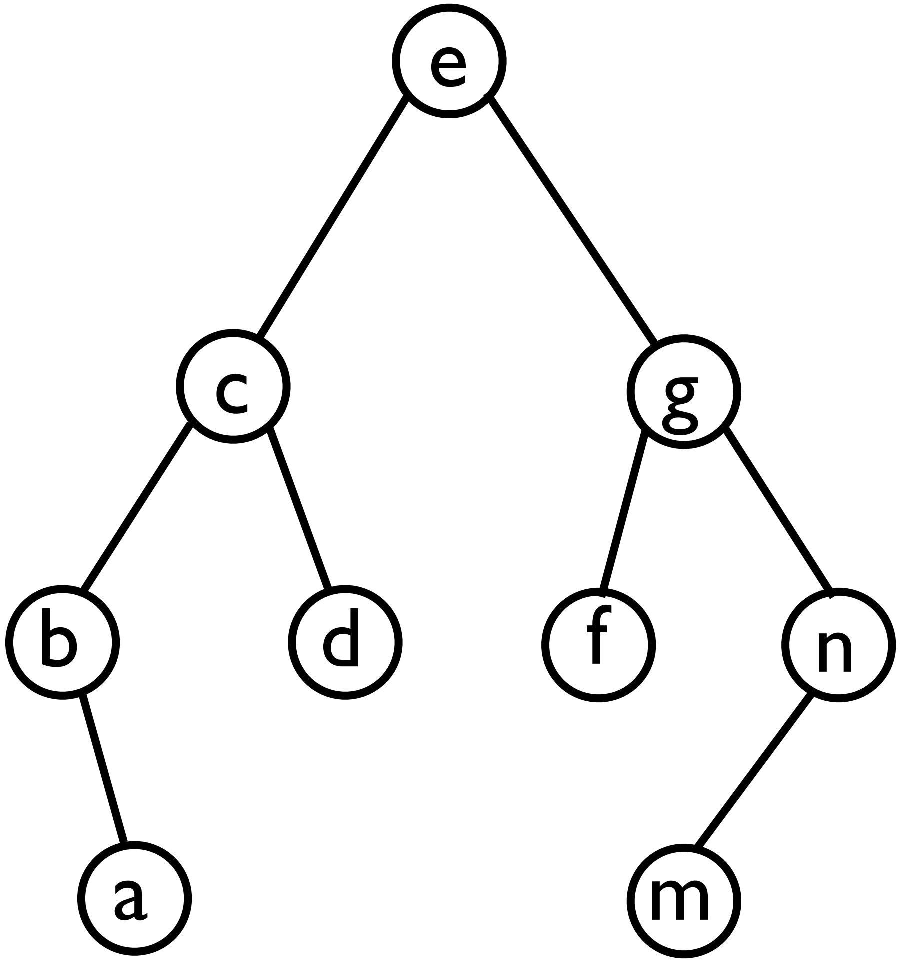


No

Binary search trees

Which of these trees are valid BSTs?



Binary search trees

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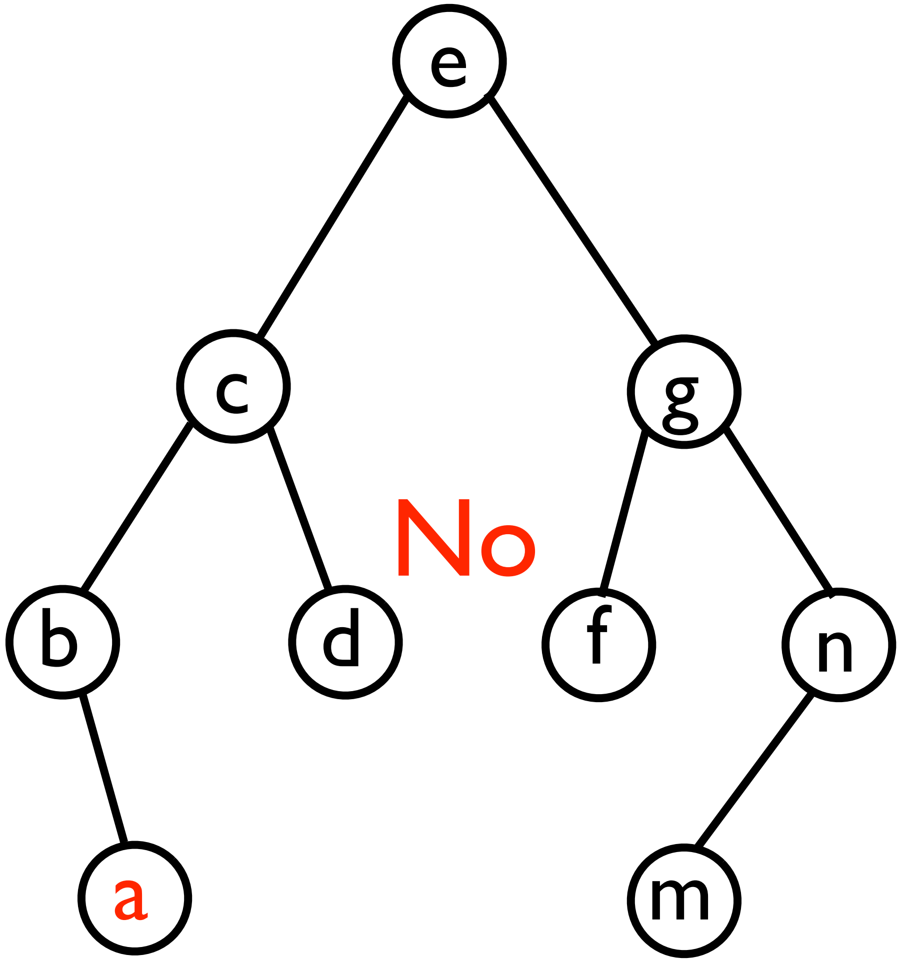


No

Binary search trees

```
class BinarySearchTree<T extends Comparable...> {
  static class Node<T> {
    T _data;
    Node<T> _leftChild, _rightChild;
  }
  Node<T> _root = null;  // BST is initially empty

  ...
}
```

Binary search trees

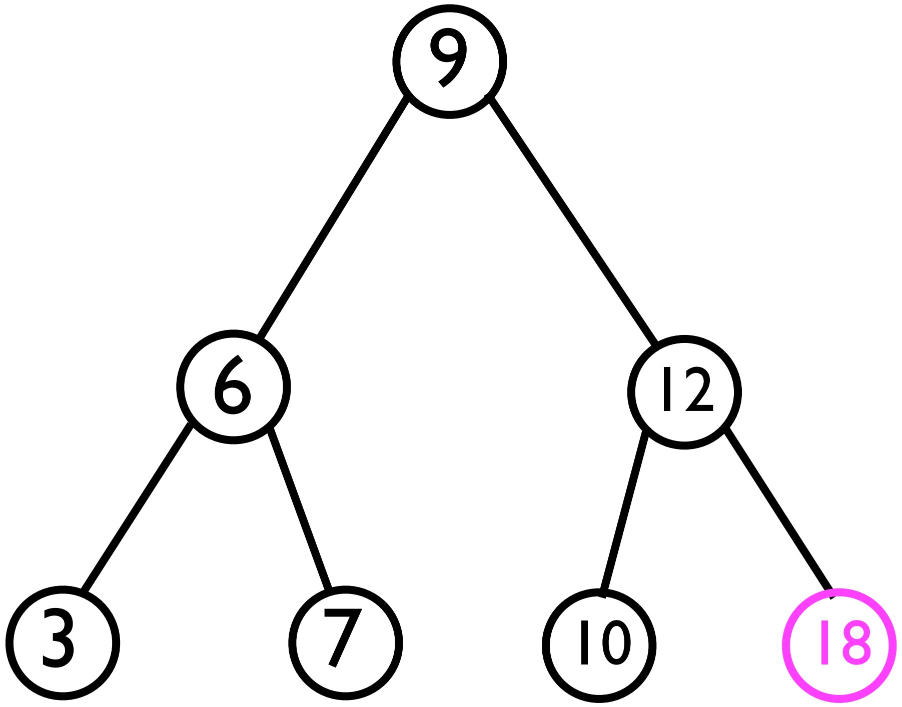
- BSTs do not permit `null` elements:
  - Unclear what "value" they should have compared to other elements.

Binary search trees

- Let us implement the following operations on BSTs:
  - `T find (T o);`
  - `T findSmallest ();`
  - `T findLargest ();`
  - `add (T o);`
  - `remove (T o);`
- To accomplish this, we will also need a few helper methods (not exposed to user):
  - `Node<T> findNode (Node<T> root, T o);`
  - `Node<T> findSuccessor (Node<T> node);`
  - `Node<T> findParent (Node<T> root, T o);`

Finding the largest element

- Due to the ordering property, finding the largest element of a BST is easy -- we just return the *right-most node* in the whole tree.



Finding the largest element

- Due to the ordering property, finding the largest element of a BST is easy -- we just return the *right-most node* in the whole tree.

```
T findLargest (Node<T> root) {

        // Iterative solution?

        // Recursive solution?

}
```

Finding the largest element

- Due to the ordering property, finding the largest element of a BST is easy -- we just return the *right-most node* in the whole tree.

Iterative solution

```
T findLargest (Node<T> root) {
  Node<T> node = root;
  while (node._rightChild != null) {
    node = node._rightChild;
  }
  return node._data;
}
```

Finding the largest element

- Due to the ordering property, finding the largest element of a BST is easy -- we just return the *right-most node* in the whole tree.

Recursive solution

```
T findLargest (Node<T> root) {
  if (root._rightChild == null) {
    return root._data;
  } else {
    return findLargest(root._rightChild);
  }
}
```

Base case

Recursive part

Finding the smallest element

- Due to the ordering property, finding the smallest element of a BST is easy -- we just return the *left-most node* in the whole tree.

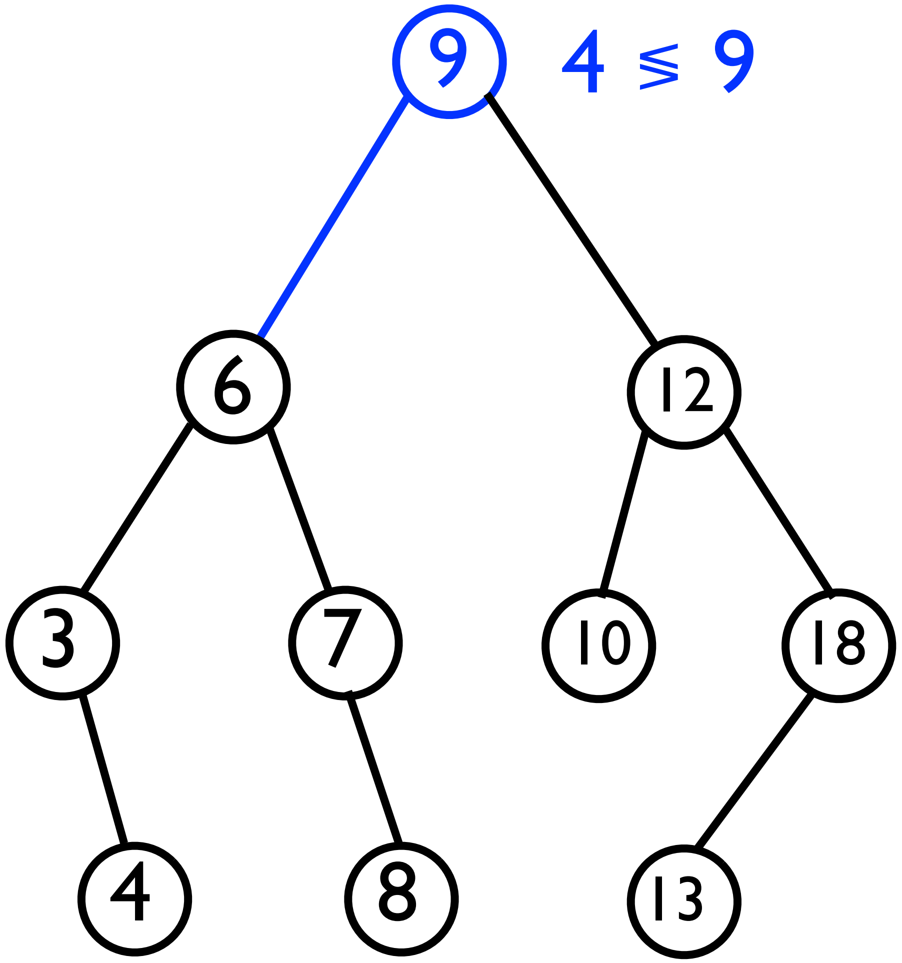
```
T findSmallest (Node<T> root) {
  Node<T> node = root;
  while (node._leftChild != null) {
    node = node._leftChild;
  }
  return node._data;
}
```

Finding a node

- The ordering property of binary search trees also enables efficient search for any *particular* node.
- Due to the ordering property, there is only one place in a given BST where value o would be stored.
    - If it's not there, then o is not contained in the BST -- hence, we return null.

Finding a node

- Given the BST below, suppose we wish to find node 4.
  - We always start at the root and recurse.



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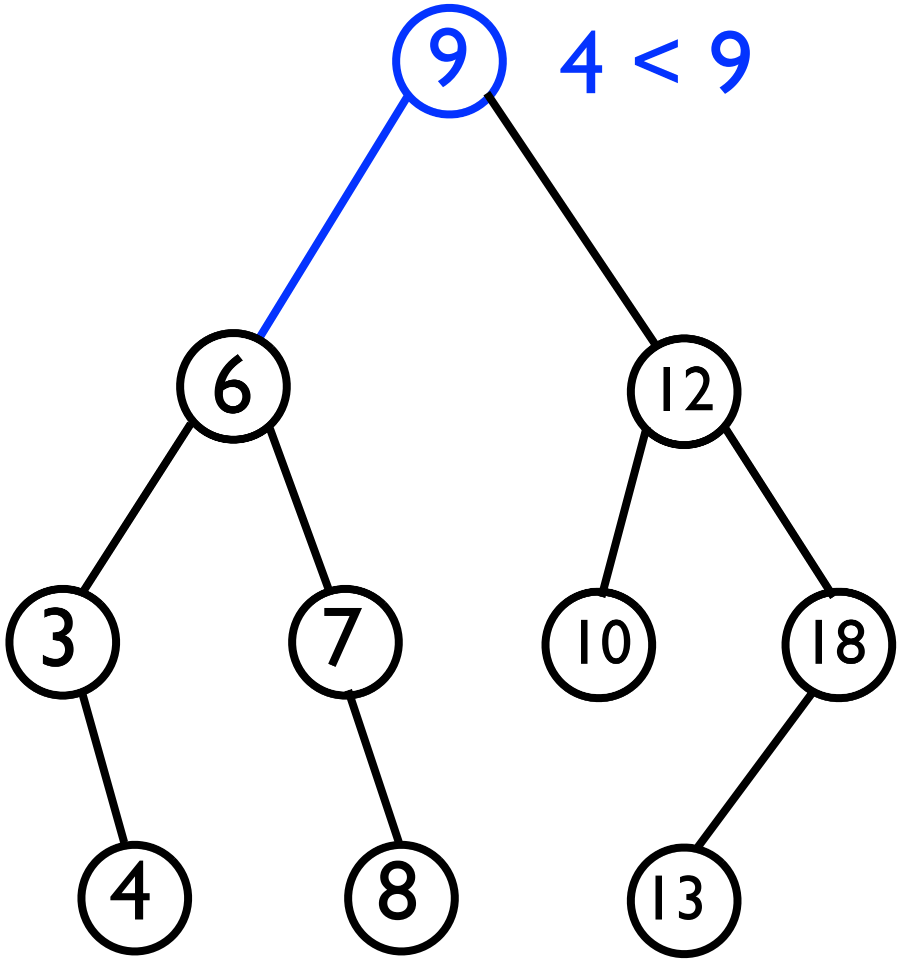

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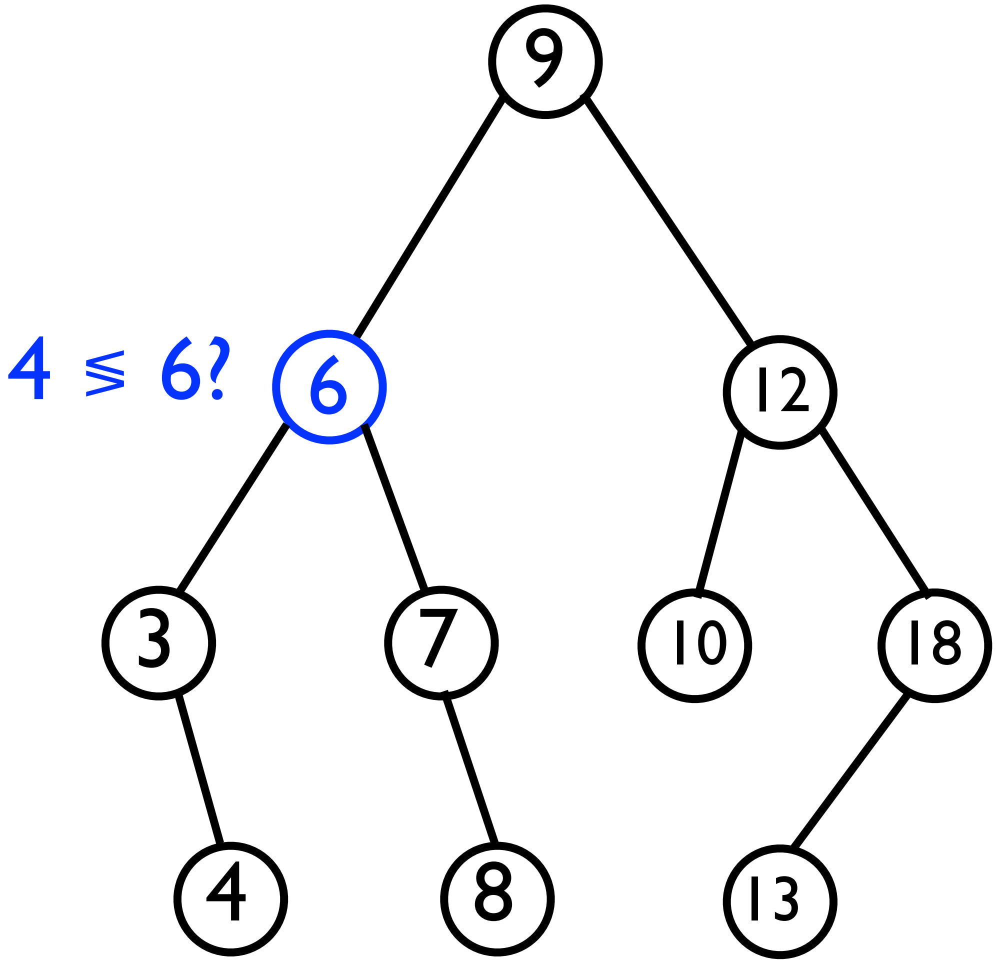

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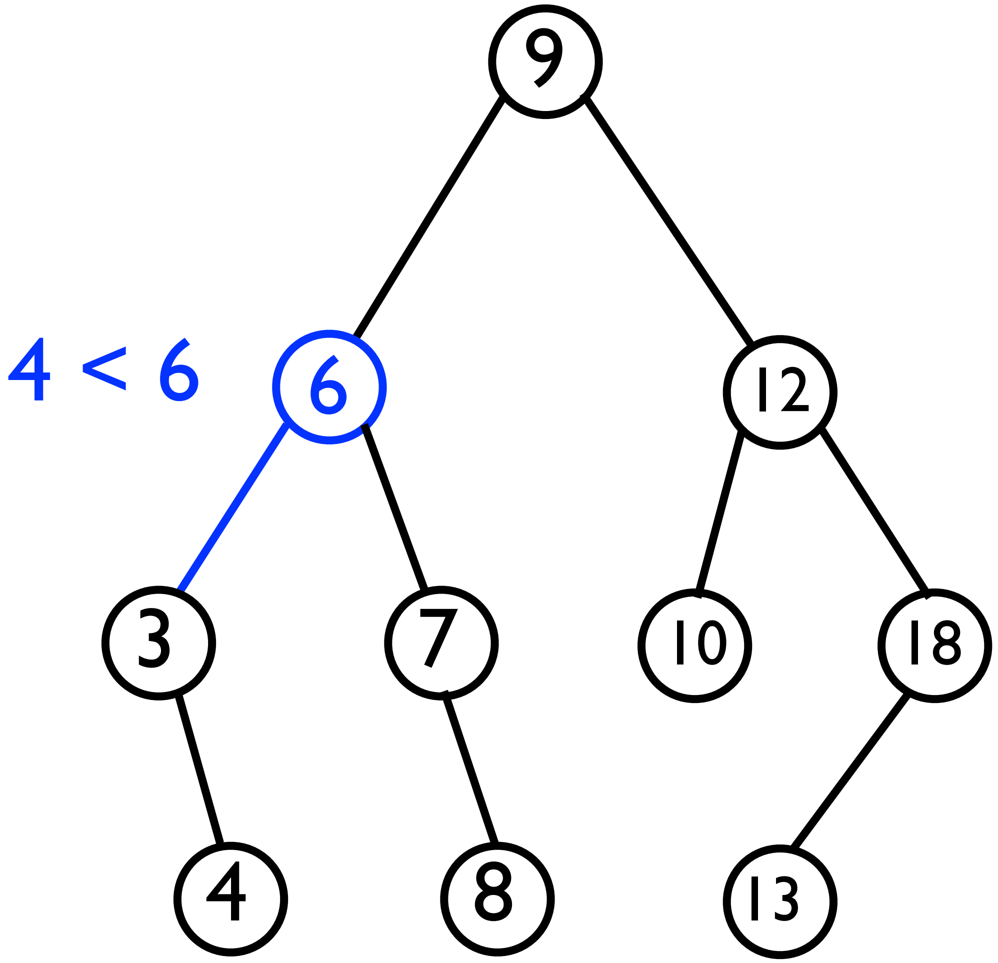

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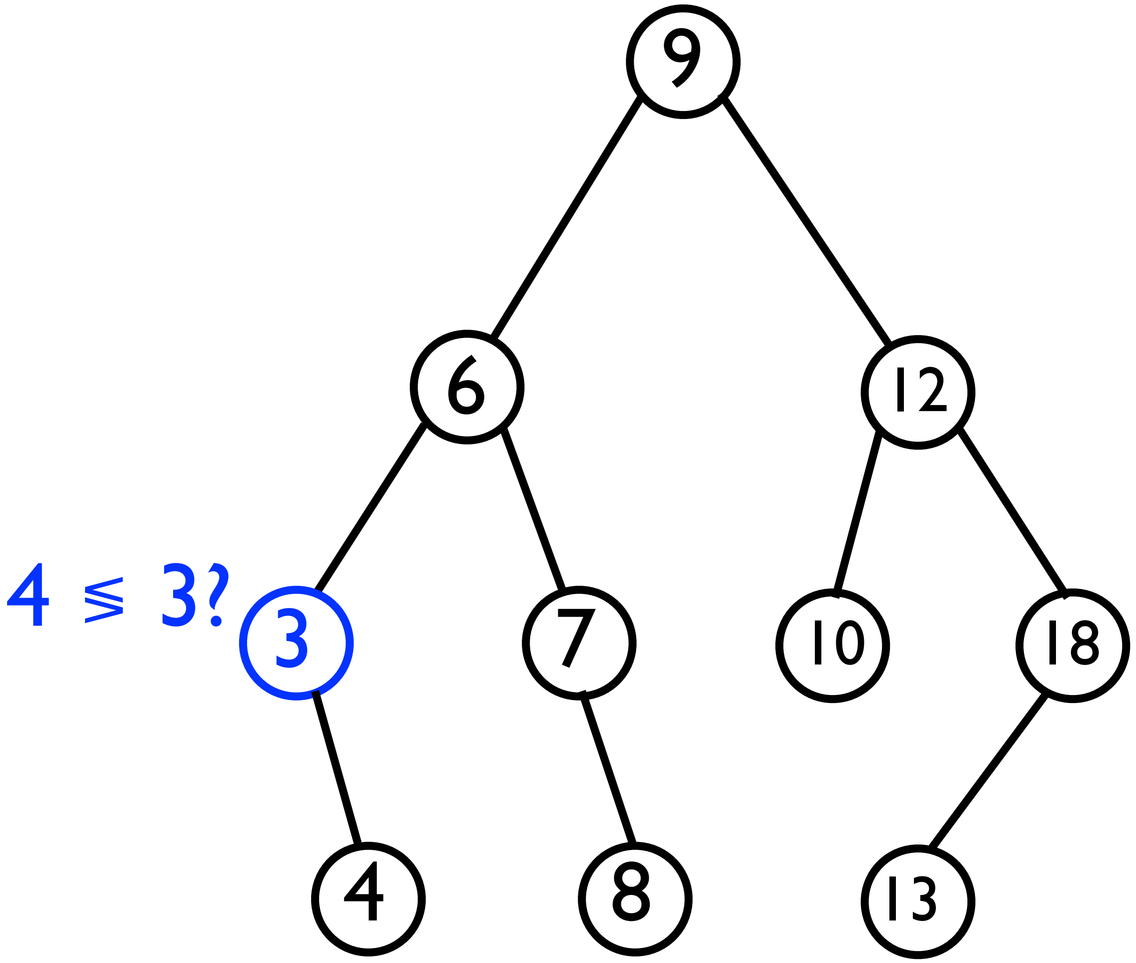

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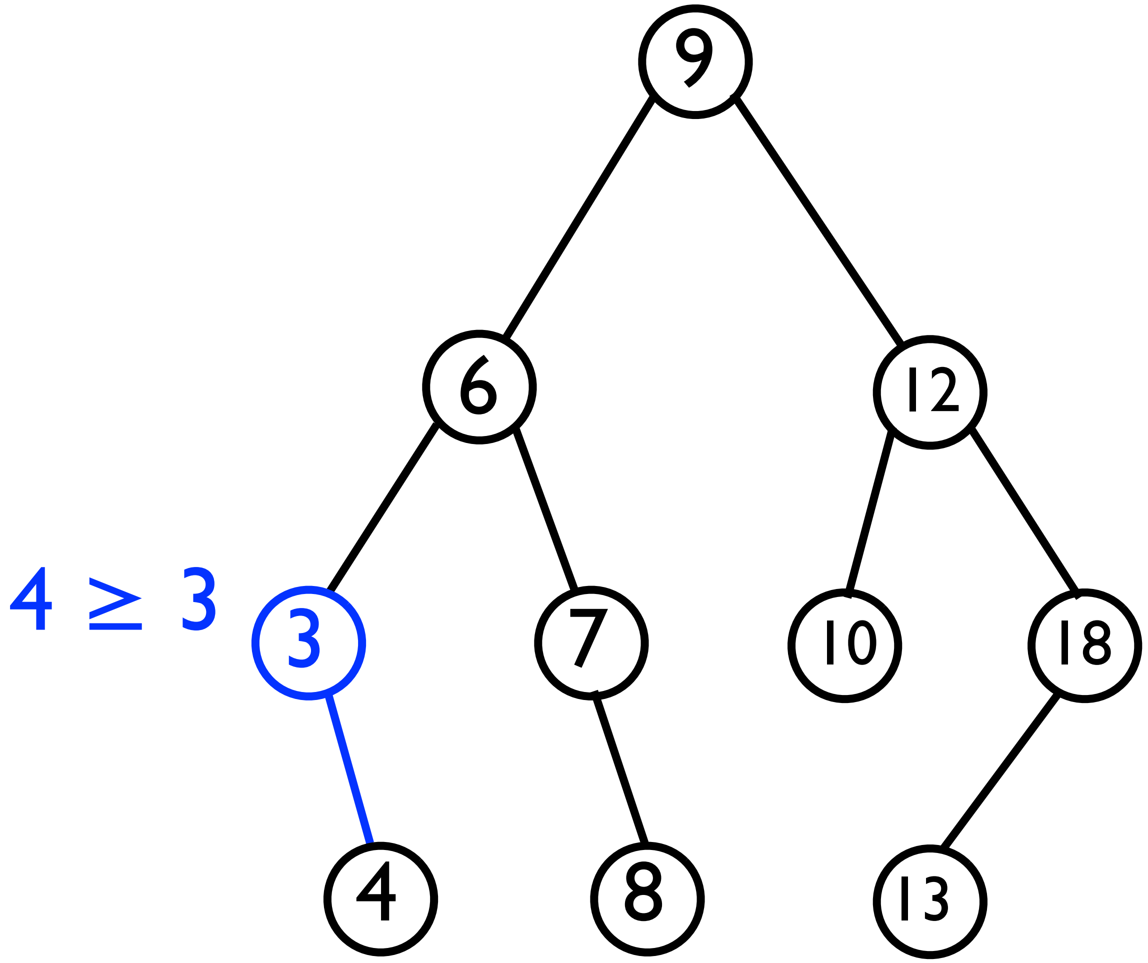

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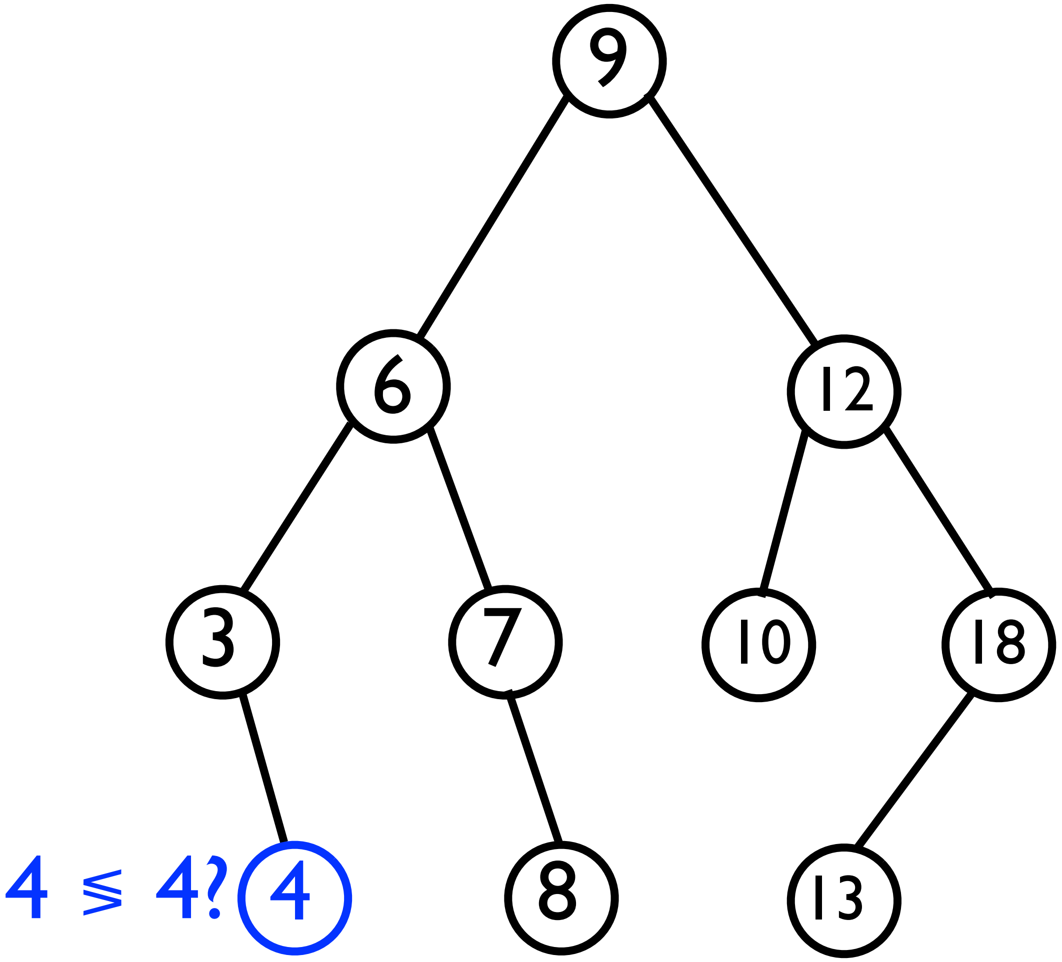

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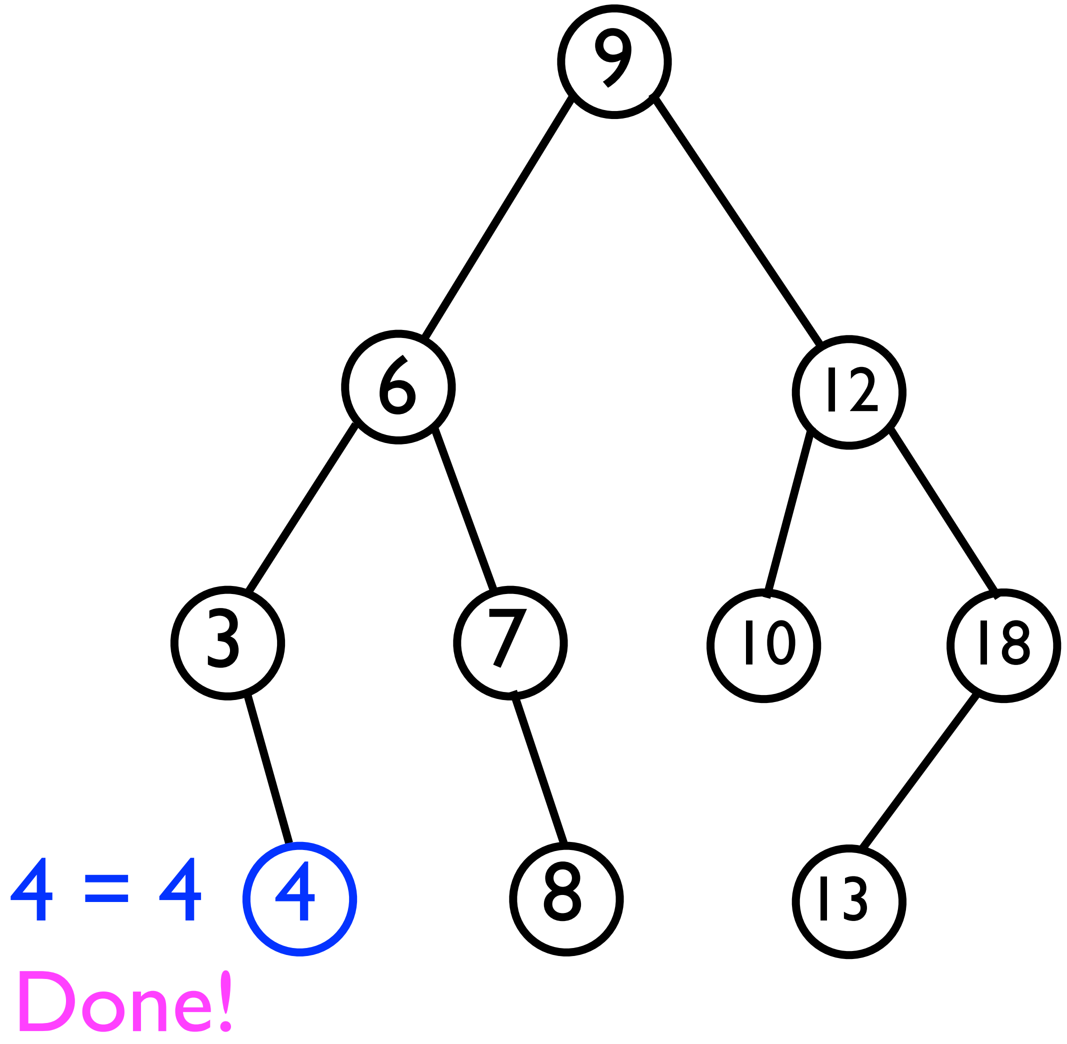


Finding a node

- Code:

```
// Returns the Node containing o, or else
// null if o is not contained in the BST.
Node<T> findNode (Node<T> root, T o) {
  if (root._data.equals(o) {
    return root;
  } else if (root._data.compareTo(o) < 0 && // Right subtree
             root._rightChild != null) {
      return findNode(root._rightChild, o);
  } else if (root._data.compareTo(o) >= 0 && // Left subtree
             root._leftChild != null) {
     return findNode(root._leftChild, o);
  } else {
    return null;
  }
}
```

Due to the ordering property, there is only *one* place in a given BST where value `o` would be stored. If it's not there, then `o` is *not contained in the BST* -- hence, we return `null`.

Finding a node

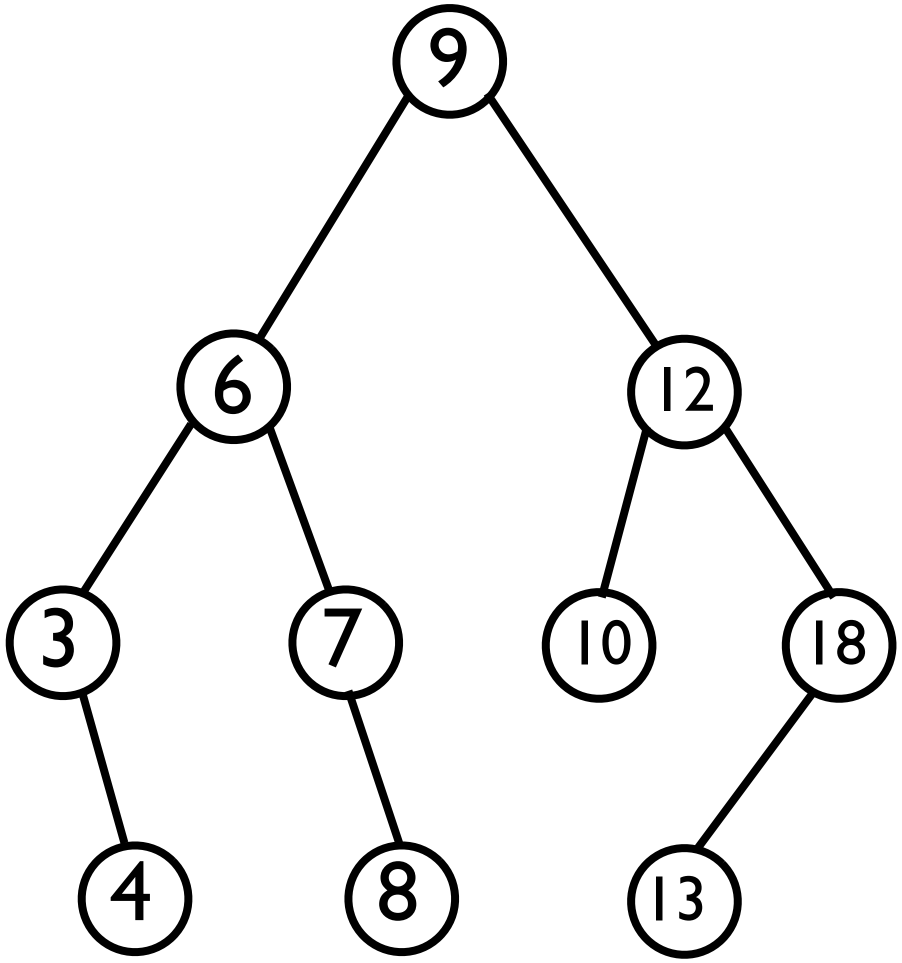
- The **`findNode(root, o)`** method would not be exposed to the user in the BinarySearchTree ADT interface.

- However, we can "wrap" this method with **`T find (T o)`** so that the underlying node infrastructure is hidden:

```
T findNode (T o) {
  if (_root == null) {
    throw NoSuchElementException();
  } else {
    final Node<T> node = findNode(_root, o);
    if (node == null) {
      throw NoSuchElementException();
    } else {
      return node._data;
  }
}
```

Finding a node's successor

- It will turn out to be useful to be able to find a node's *successor* in the BST.
  - The *successor* of node *n* is the node with the *next higher value*.



Finding a node’s successor

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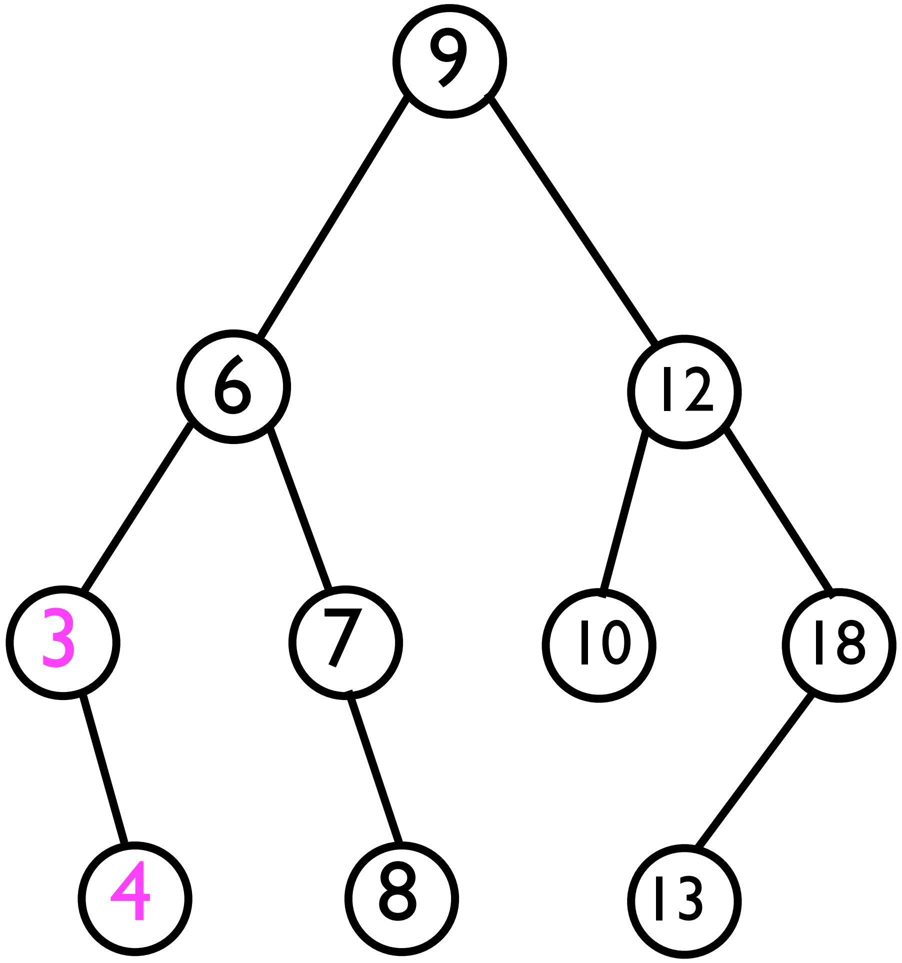
  - The *successor* of node *n* is the node with the *next higher value*.
  - Examples:
    Successor of 3 is 4.
    Successor or 4 is 6.
    Successor of 12 is 13.
    Successor of 8 is 9.



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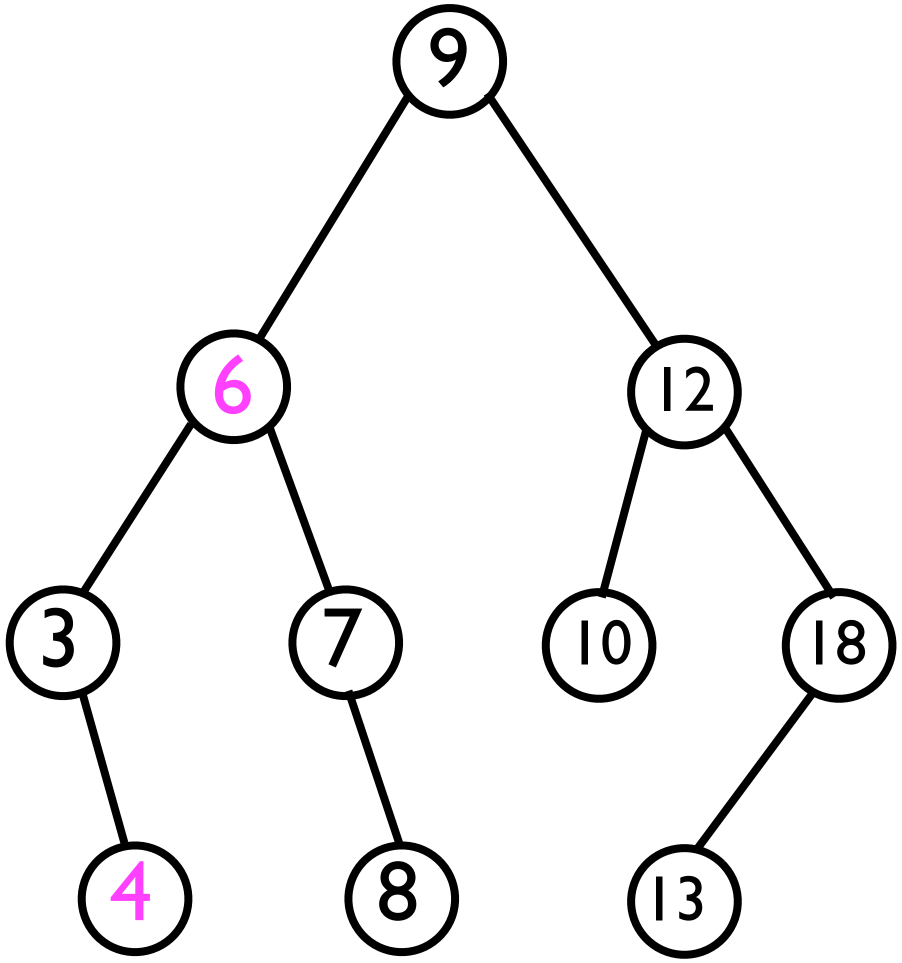

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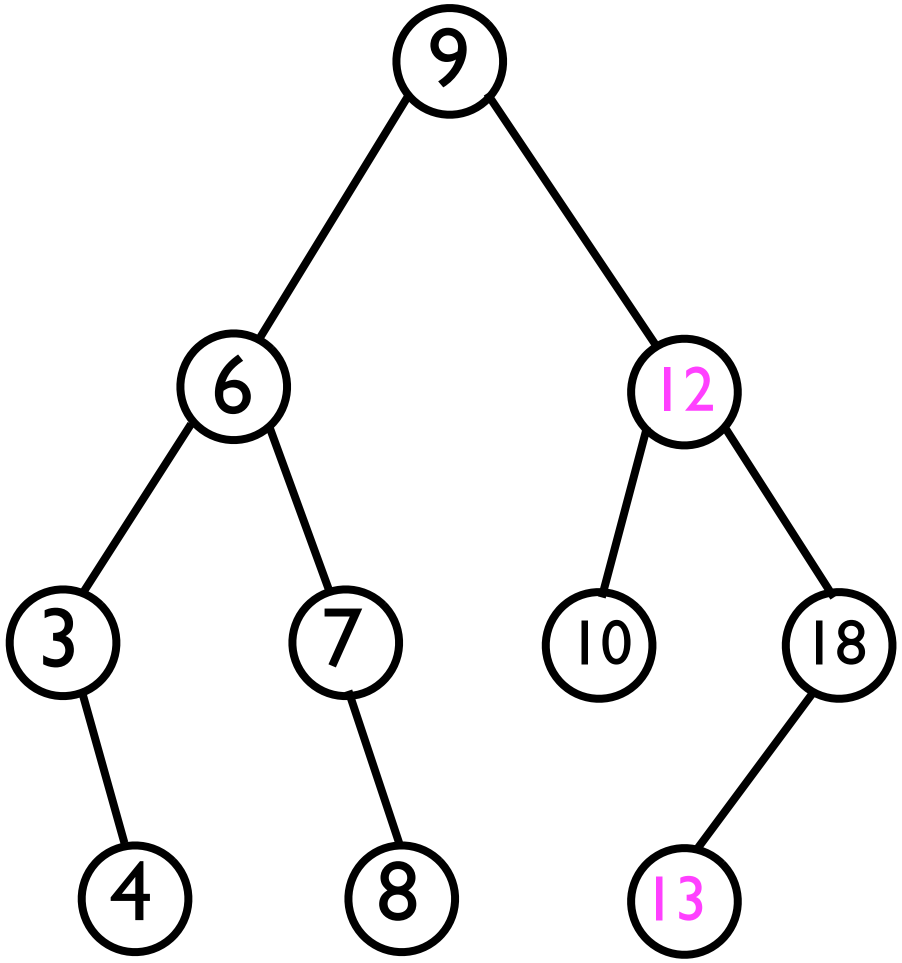

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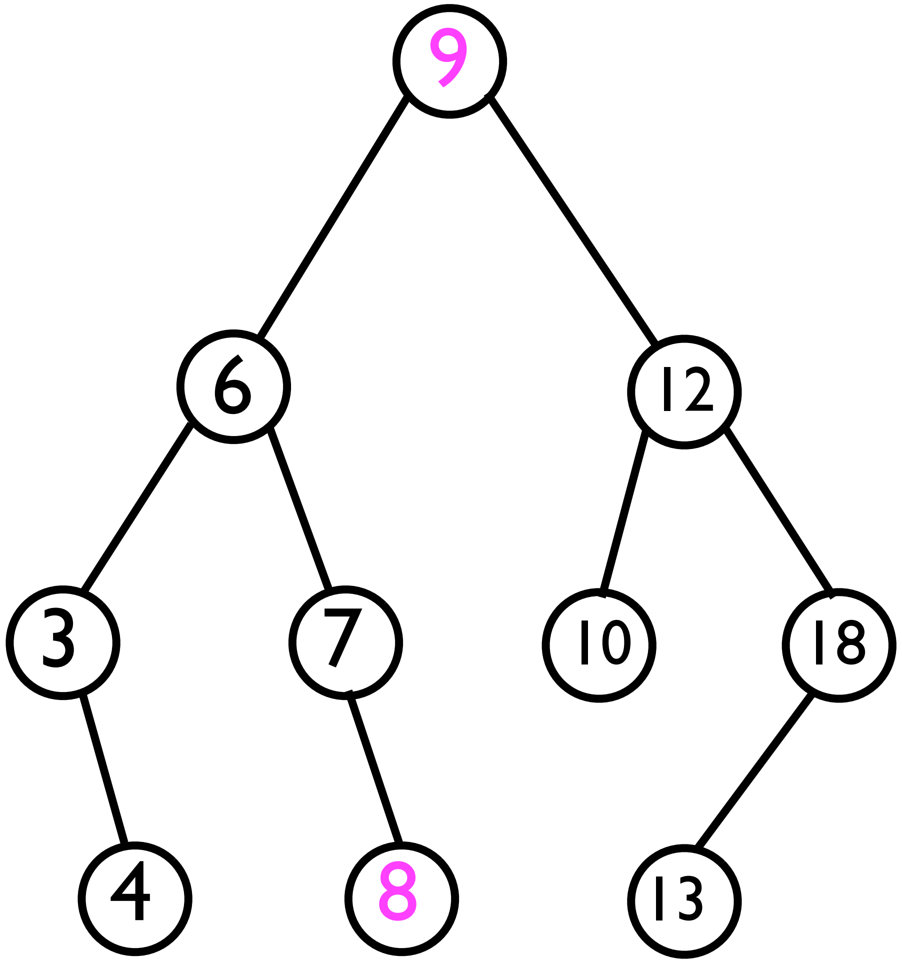


Finding a node's successor

- A *successor* node of *n* -- if it exists -- is found by *either*:

  1. Descending into *n*'s right sub-tree, and then recursively selecting left-child until no left child exists.

     - *Intuition*: The right sub-tree has values bigger than n; we want the smallest such value (left-most node).

  2. Finding the *lowest* ancestor of *n* whose left child is also an ancestor of *n*.

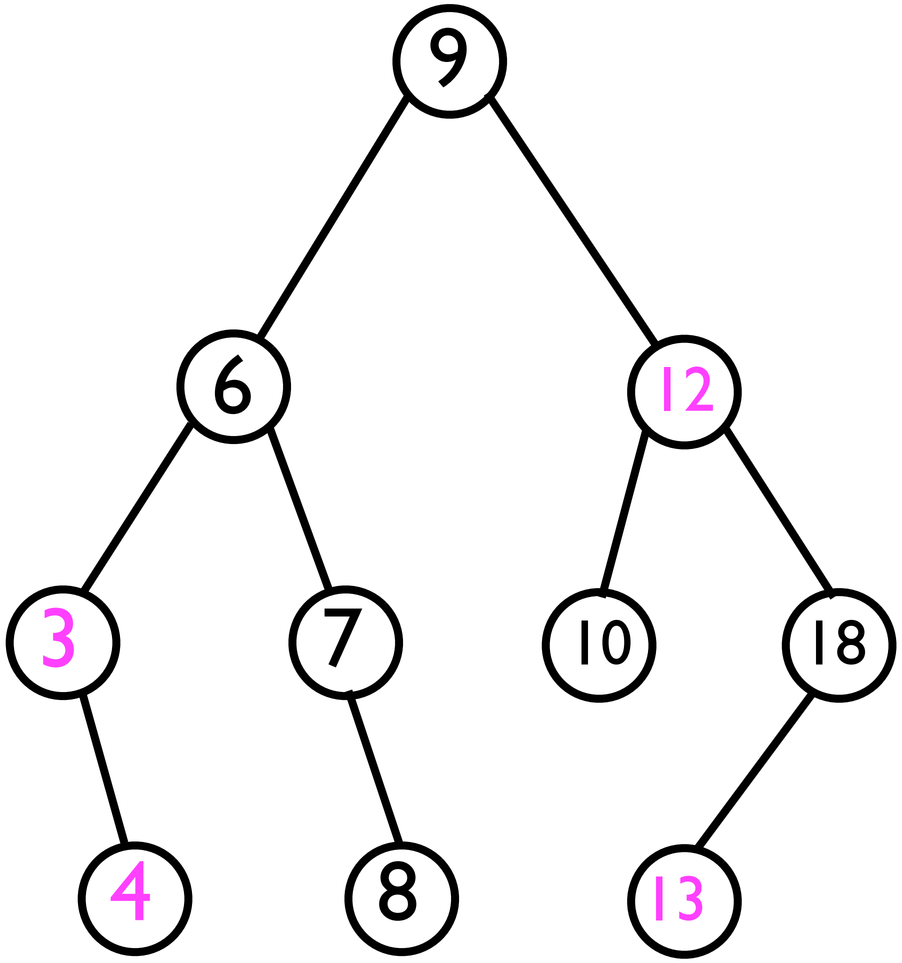
     - *Intuition*: Move "up-and-left" in the BST until we can finally "move right" again, i.e., *towards a higher valued node*.

Finding a node's successor

- A *successor* node of *n* -- if it exists -- is found by *either*:

1. Descending into *n*'s right sub-tree, and then recursively selecting left-child until no left child exists.
2. Finding the *lowest* ancestor of *n* whose left child is also an ancestor of *n*.

- Examples:
  Successor of 3 is 4.
  Successor or 4 is 6.
  Successor of 12 is 13.
  Successor of 8 is 9.



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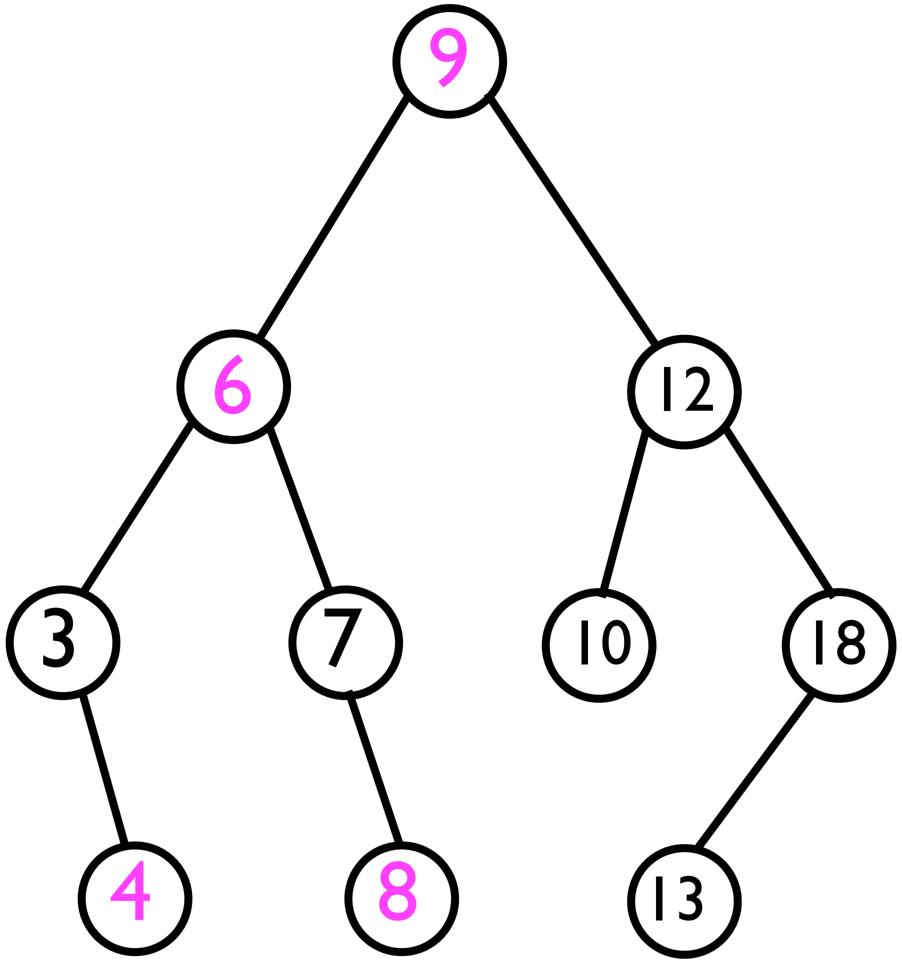


Finding a node's successor

- The code for **`Node<T> findSuccessorNode (Node<T> node)`** will be left as an "exercise for the reader".

Adding a new node

- To add a new node, we must distinguish two cases:

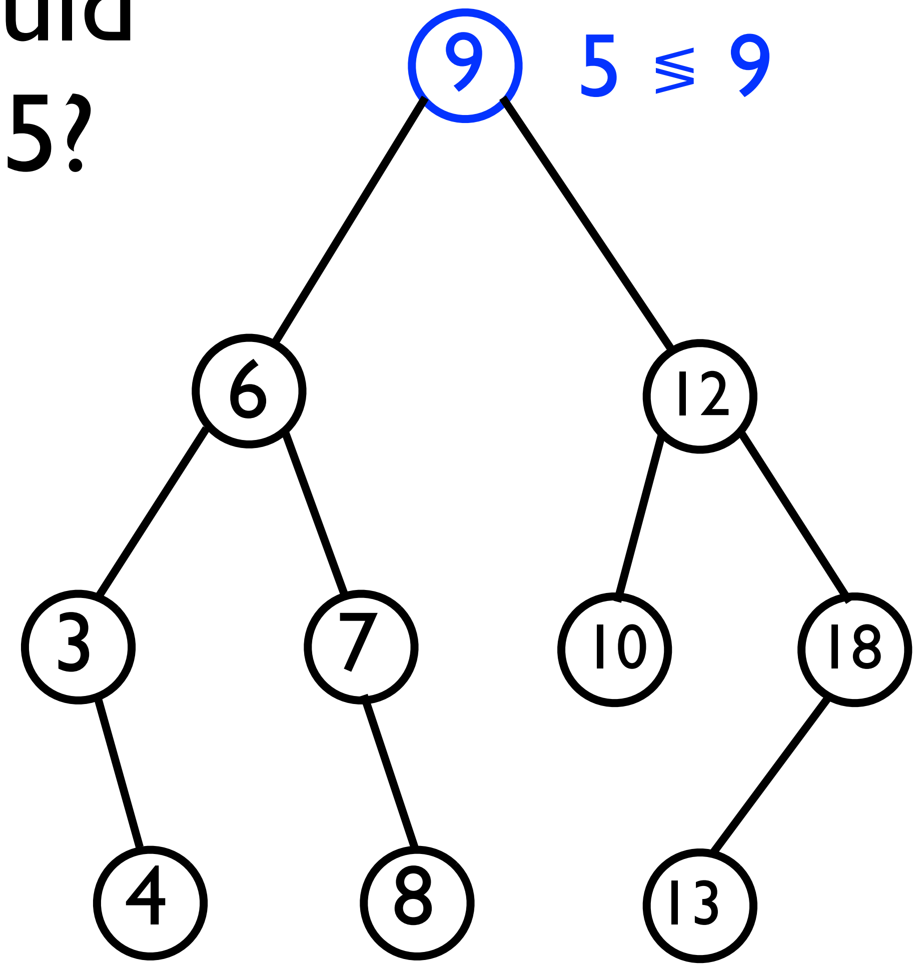
1. The new node is the *first* node in the BST.
    - In this case, we simply set this node to be the root.
2. The new node is *not* the first node in the BST.
    - Then we must find the *parent* node of the node we're about to add.
    - We then add the new node as a child of the parent.

Finding the parent of a new node

- To find the parent node of the new node *n* we want to add:
  - Recursively search from root down towards the leaf nodes, *as if node* n *were already inserted*.
  - Eventually, while recursing at node *p*, the search for the node would take us to a left/right child *that does not yet exist.*
    - At that point, we know *p* is the parent of *n*.
    - *p* is the "natural insertion point" for *n*.

Finding the parent of a new node

Where would we insert 5?



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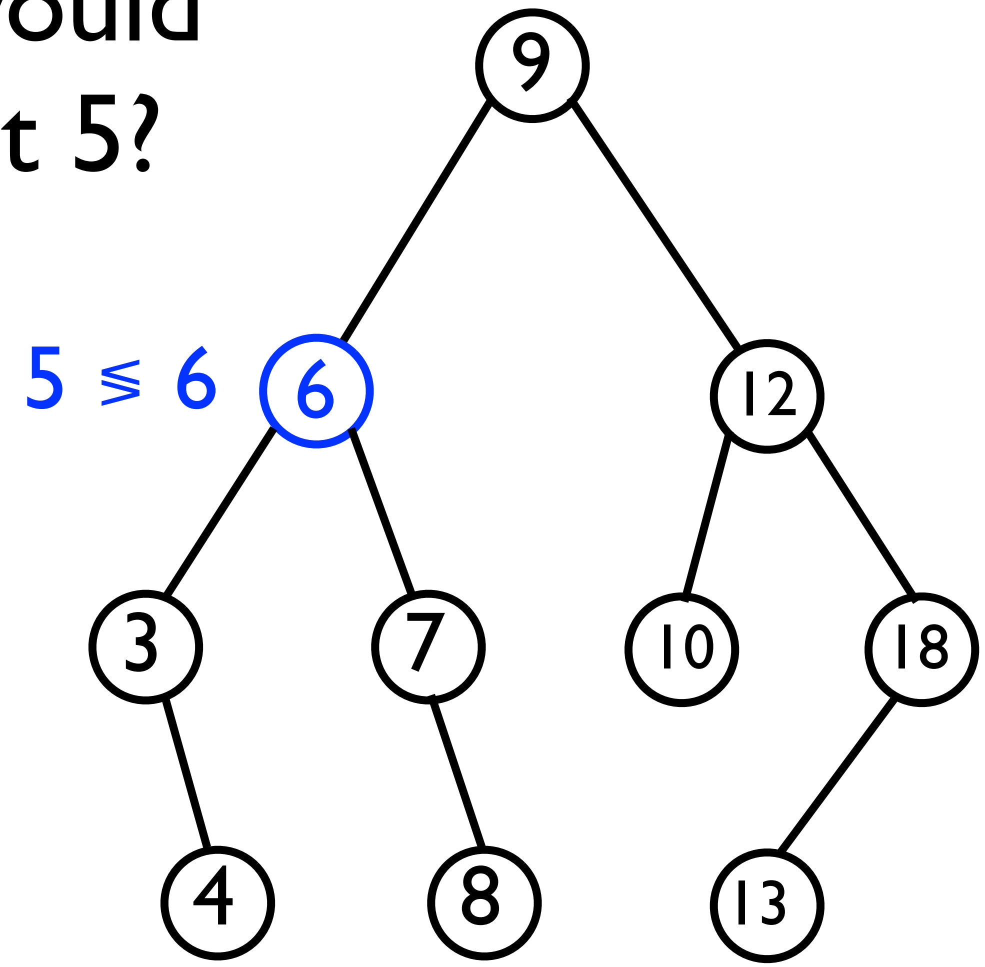

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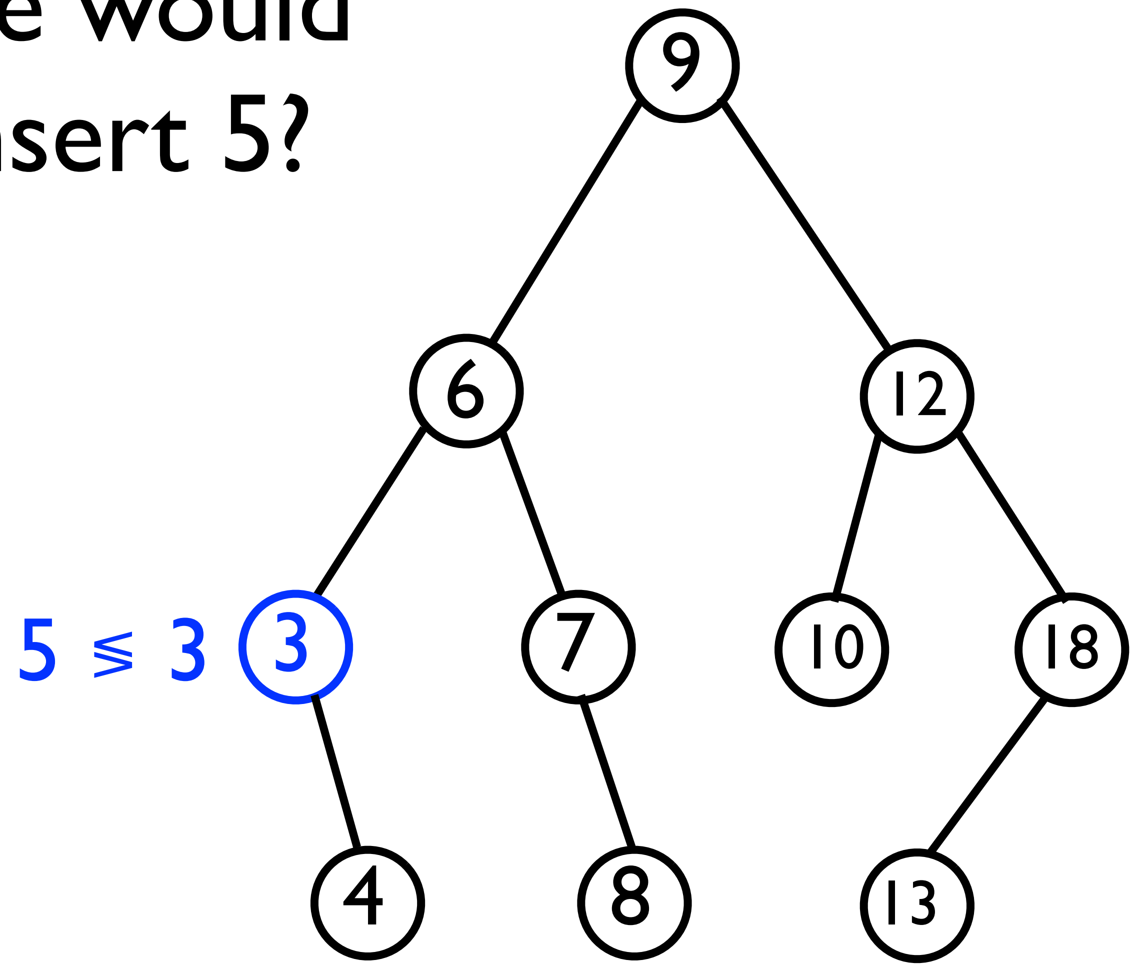

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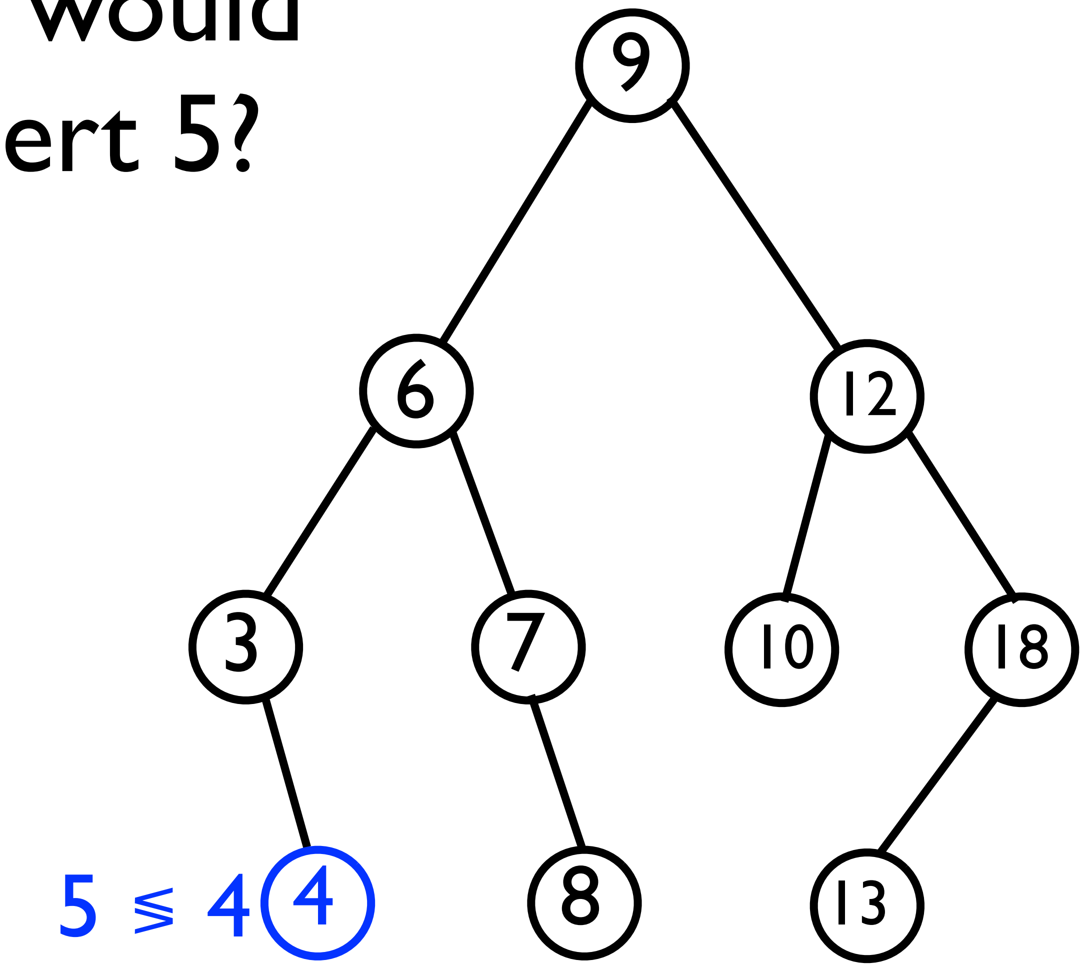

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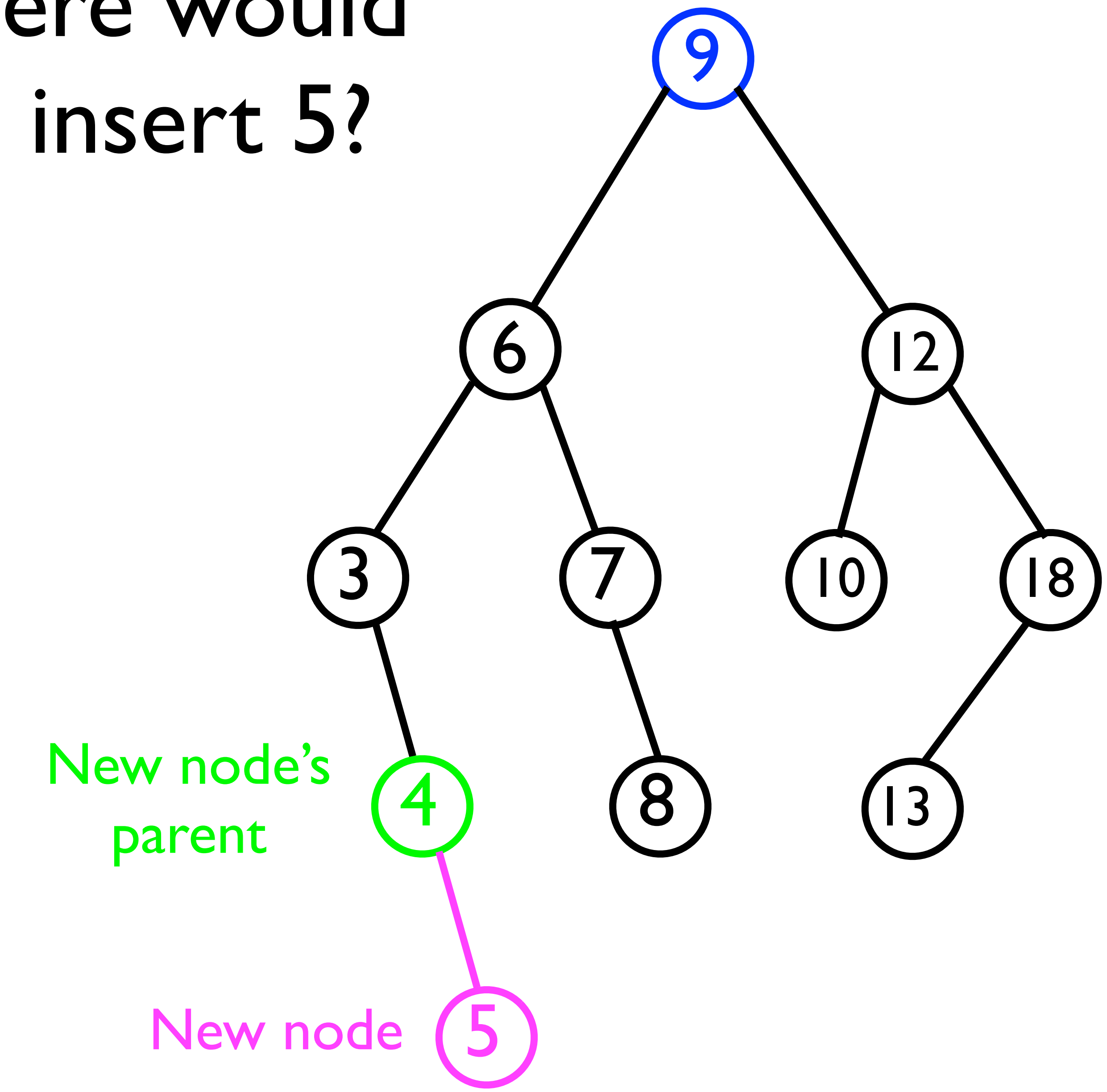


Finding the parent of a new node

```
// Searches from root for the parent node to which the
// specified new node should be added.
Node<T> findParentNode (Node<T> root, T o) {
  // Save comparison result
  final int comparison = root._data.compareTo(o);

  if (comparison < 0 && root._rightChild != null) {
    return findParentNode(root._rightChild, o);
  } else if (comparison >= 0 && root._leftChild != null) {
    return findParentNode(root._leftChild, o);
  } else {  // The appropriate left/child does not yet exist
    return root;  // Hence, we've found the parent
  }
}
```

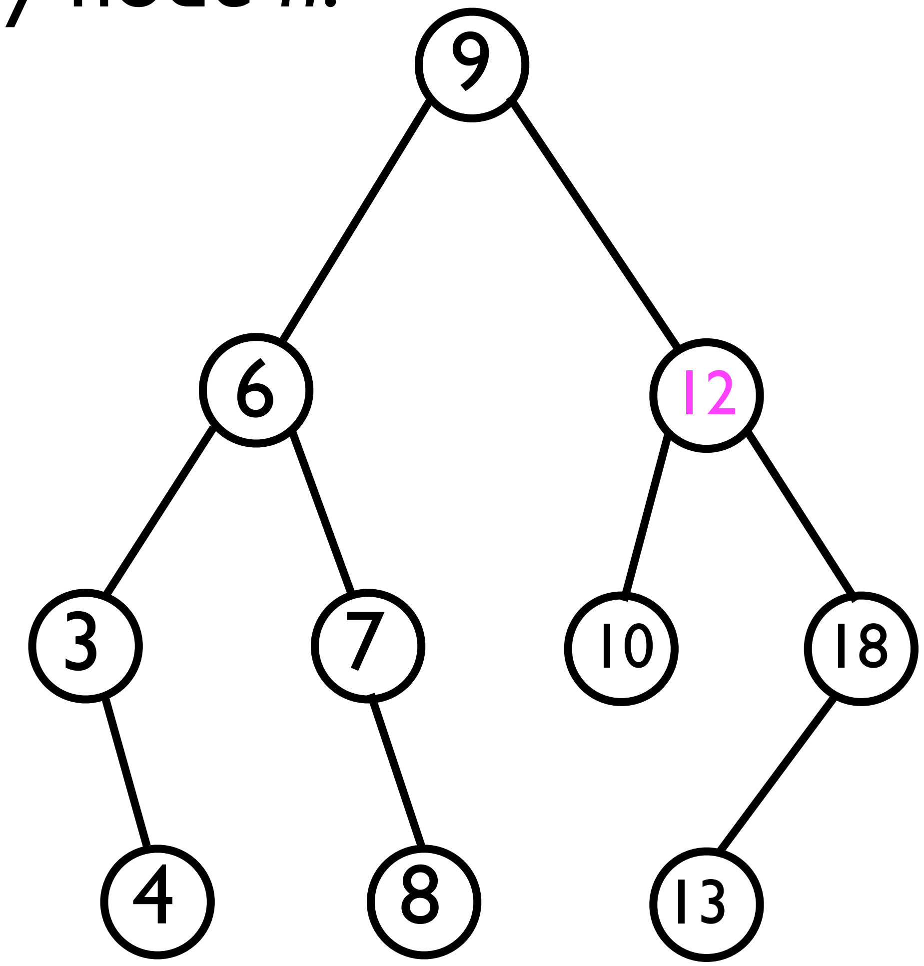
Adding a new node

- We can now implement the `add(o)` method:

```
void add (T o) {
  final Node<T> node = new Node<T>();
  node._data = o;
  if (_root == null) {  // Case 1
    _root = node;
  } else {              // Case 2
    final Node<T> parent = findParent(_root, o);
    if (parent._data.compareTo(o) < 0) {
      parent._rightChild = node;
    } else {
      parent._leftChild = node;
    }
  }
}
```

Removing a node

- When removing a node *n* from the BST, we must ensure that:
  - The resulting tree is still *connected*.
  - The resulting tree still has the *ordering property*.
- Consider what might "go wrong" when removing an arbitrary node *n*:



If we remove node 12, then we *sever its left and right sub-trees* from the rest of the BST.

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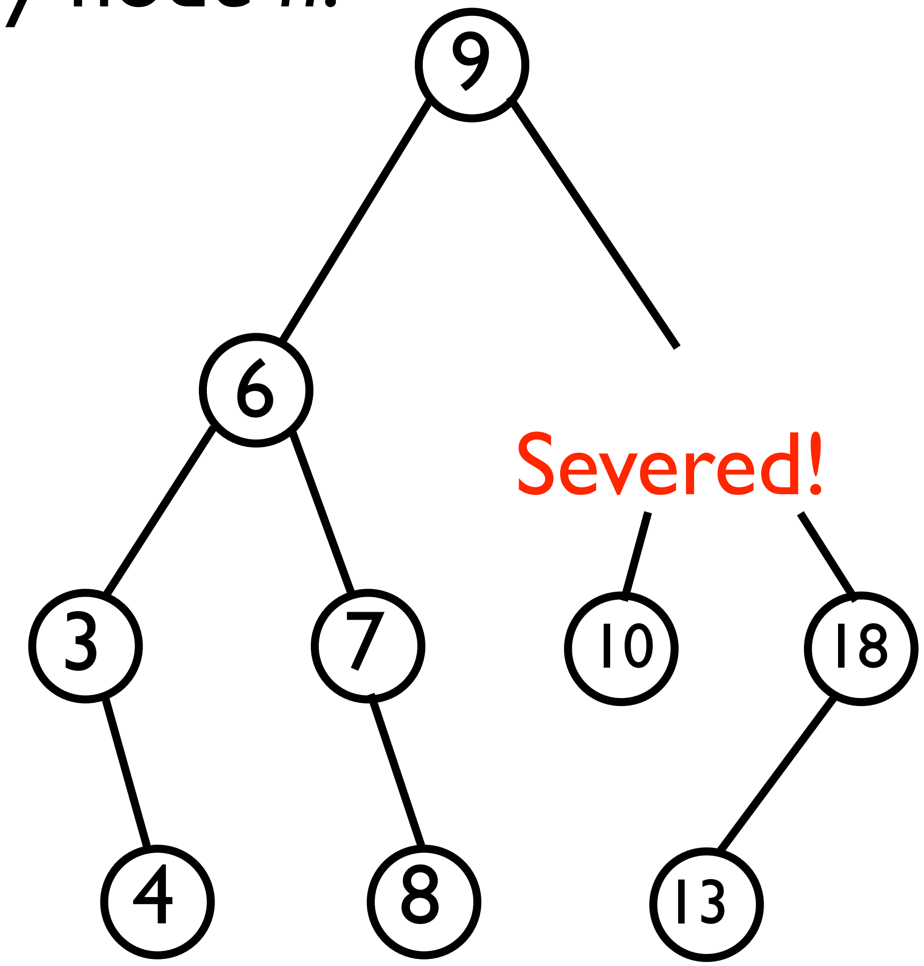

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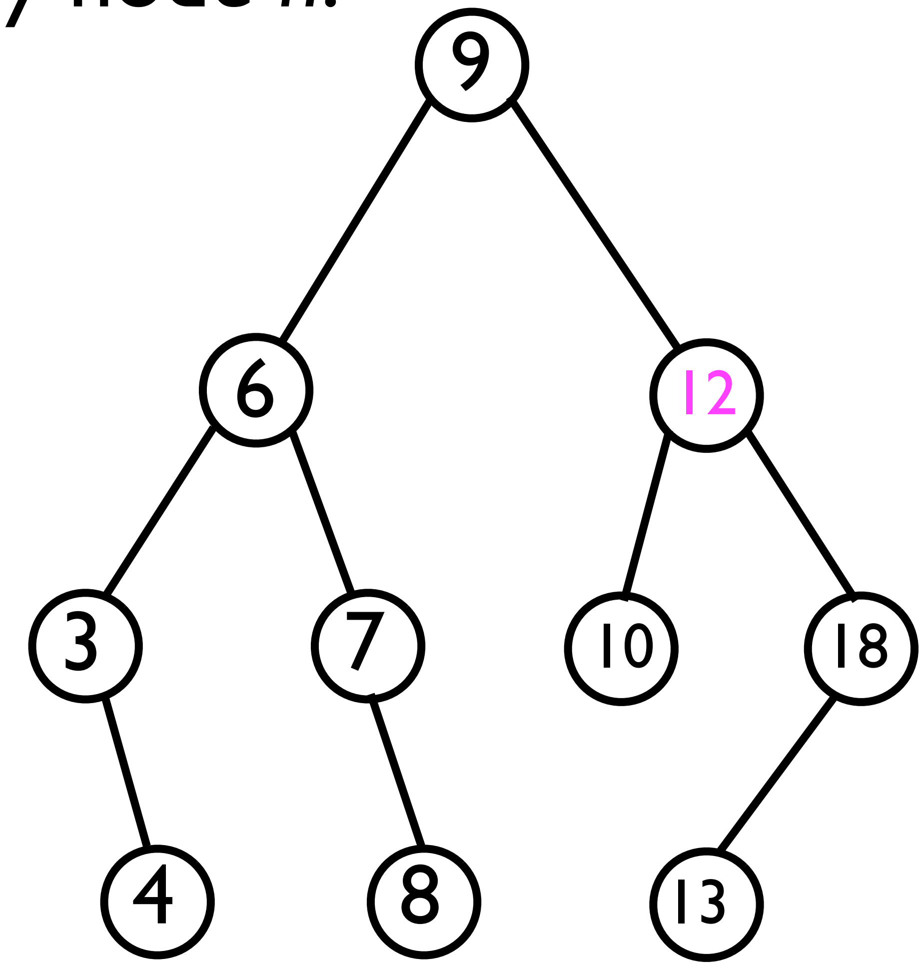

If instead we replace *n* with another node and "reconnect" another branch, we might *violate the ordering property*.

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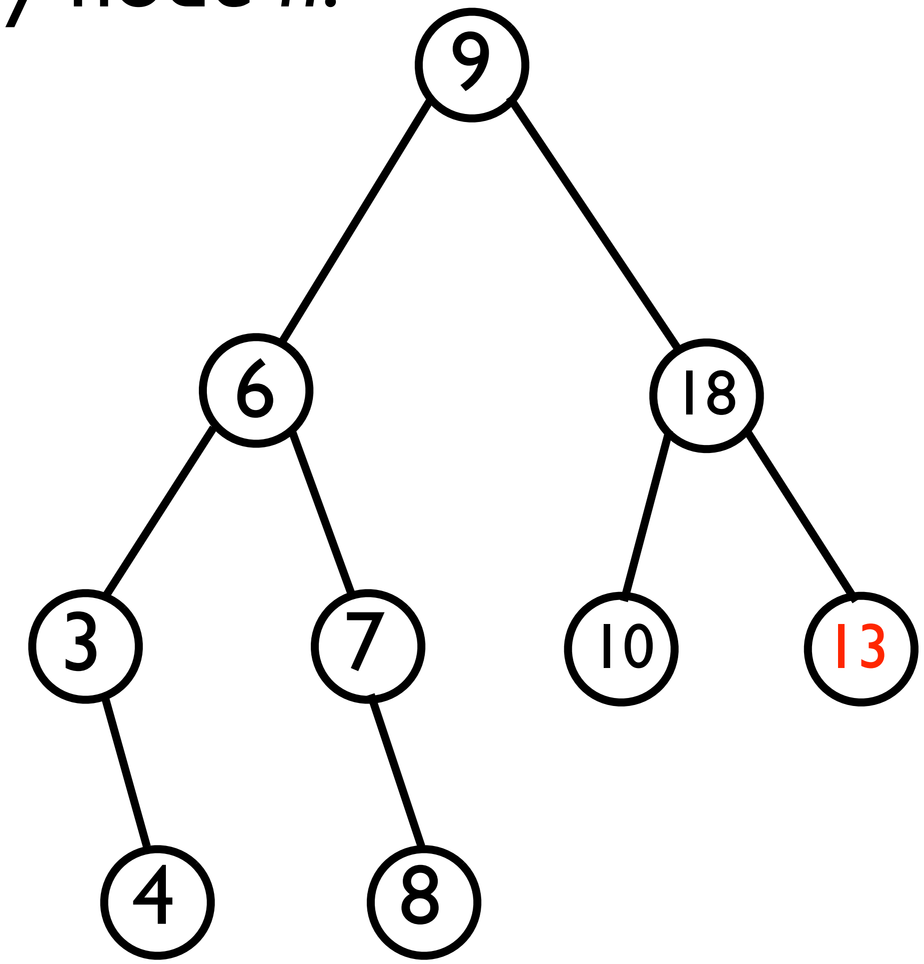


If instead we replace *n* with another node and "reconnect" another branch, we might *violate the ordering property.*

Ordering property is now violated!

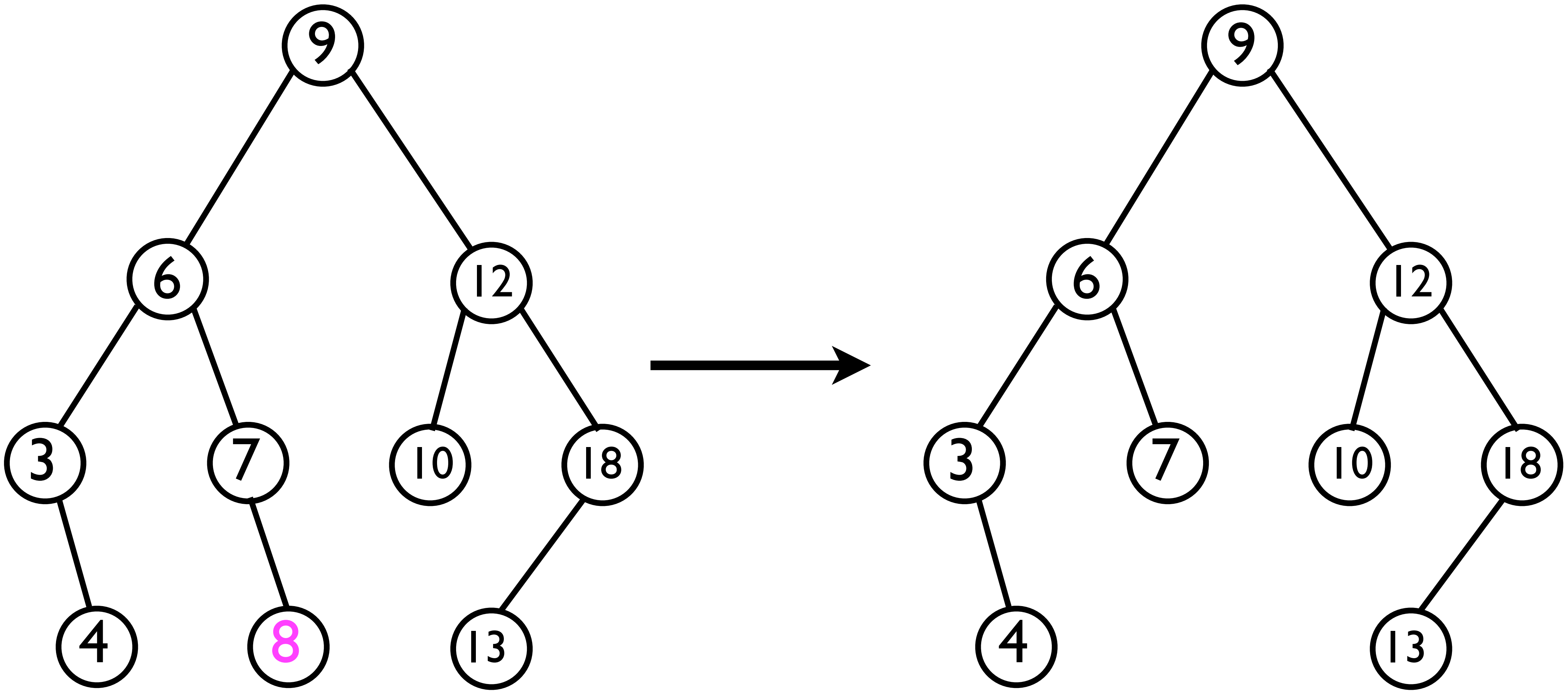
Removing a node

- To remove a node and still ensure the resulting tree is a proper BST, we must distinguish three cases:

1. *n* is a leaf node -- in this case, we just snip it off.
2. *n* is an internal node with only one child.
    - We remove *n* and "splice around" it.
3. *n* is an internal node with two child nodes.
    - We replace *n* with the value of its successor *s*, and then *recursively* remove *s*.

Removing a leaf node

Example: `bst.remove(8);`

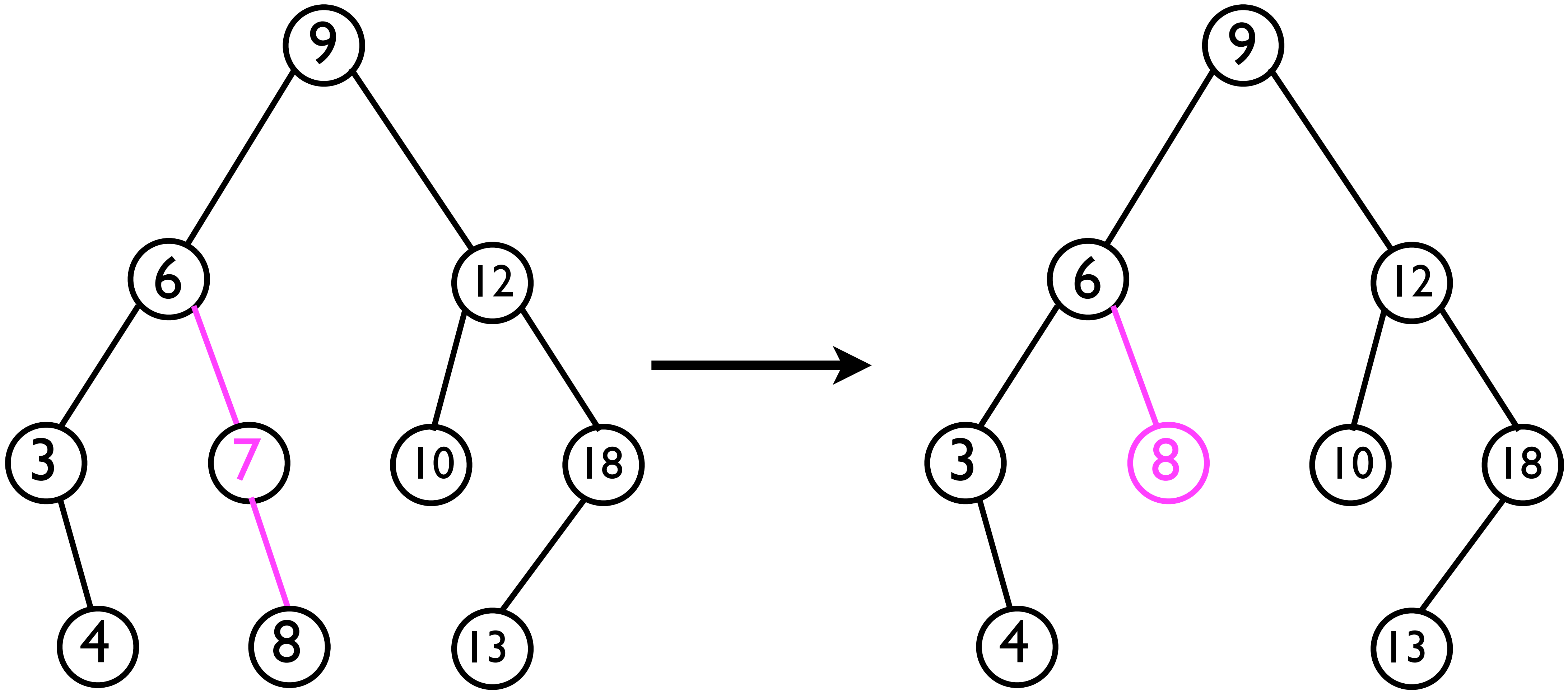


Just snip it off.

Result: We still have a BST with the ordering property preserved.

Removing a node with one child node

Example: `bst.remove(7);`

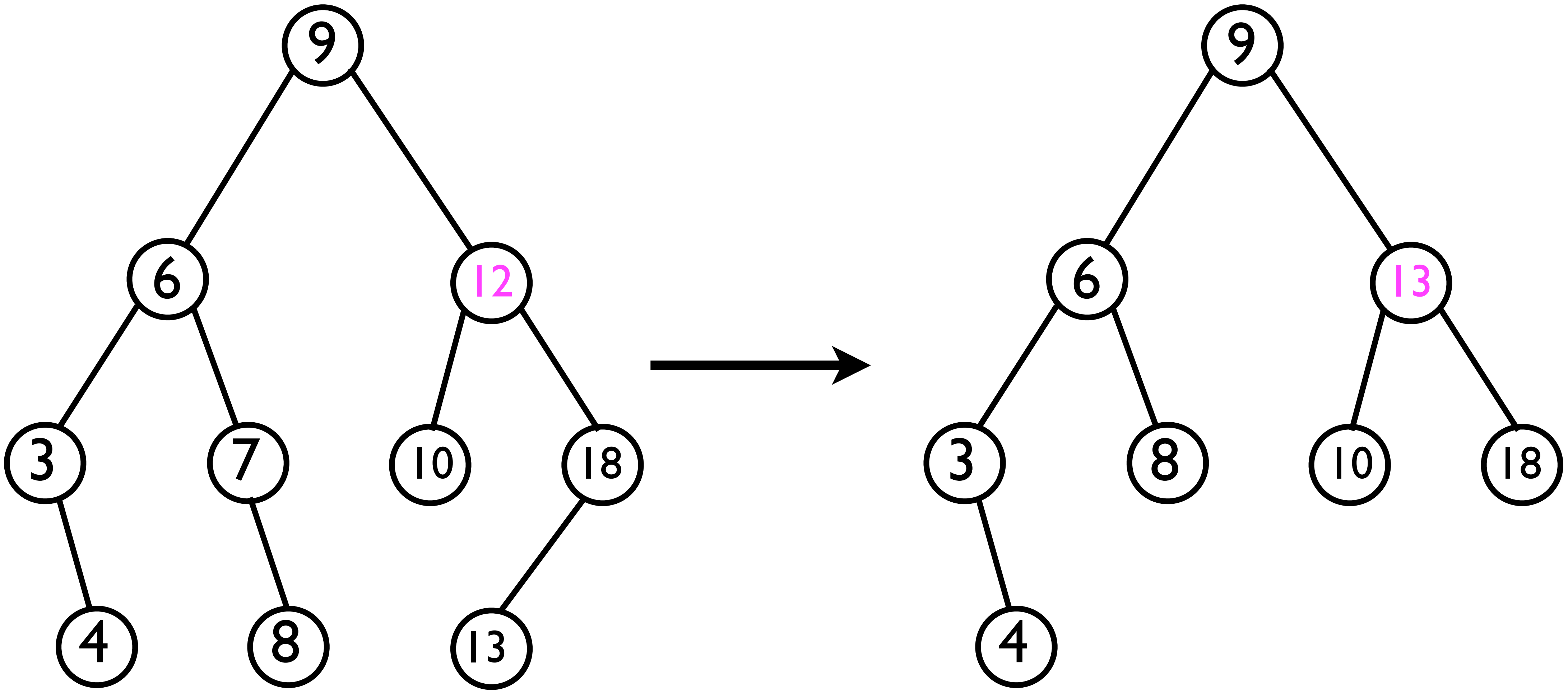


"Splice around" node 7.

Result: We still have a BST with the ordering property preserved.

Removing a node with two child nodes

Example: `bst.remove(12);`



Replace 12 with the value of its *successor*; then remove the successor node.

Result: We still have a BST with the ordering property preserved.

Removing the successor

- When removing a node *n* with two children, we replace n with the value of its successor *s*, and then remove s itself.
  - But what if *s also* has two children; then we need to remove *its* successor, and so on.
  - Will the "removal" process ever terminate?
    - Yes -- if *n* has two children, then its successor *s cannot* have a left-child. Why?

Removing the successor

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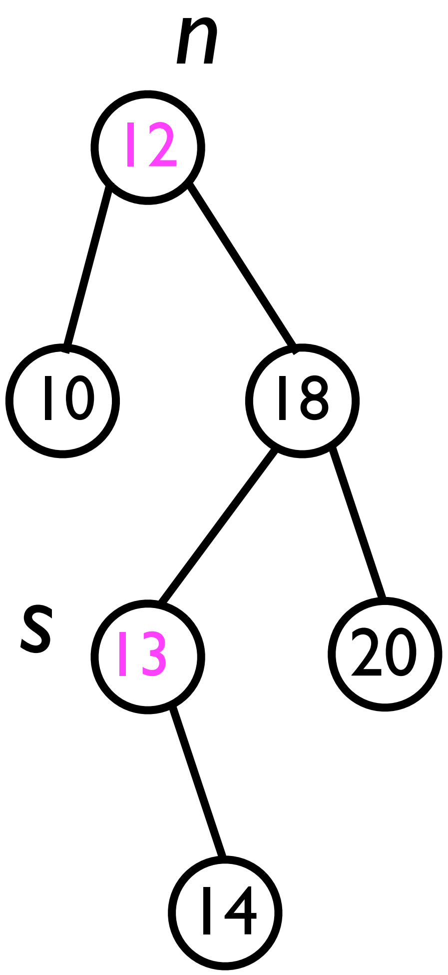
  - But what if s *also* has two children; then we need to remove *its* successor, and so on.
  - Will the "removal" process ever terminate?
    - Yes -- if *n* has two children, then its successor s *cannot* have a left-child. Why?
      - If it did, s's *that left child* would be n's successor, and not s itself.

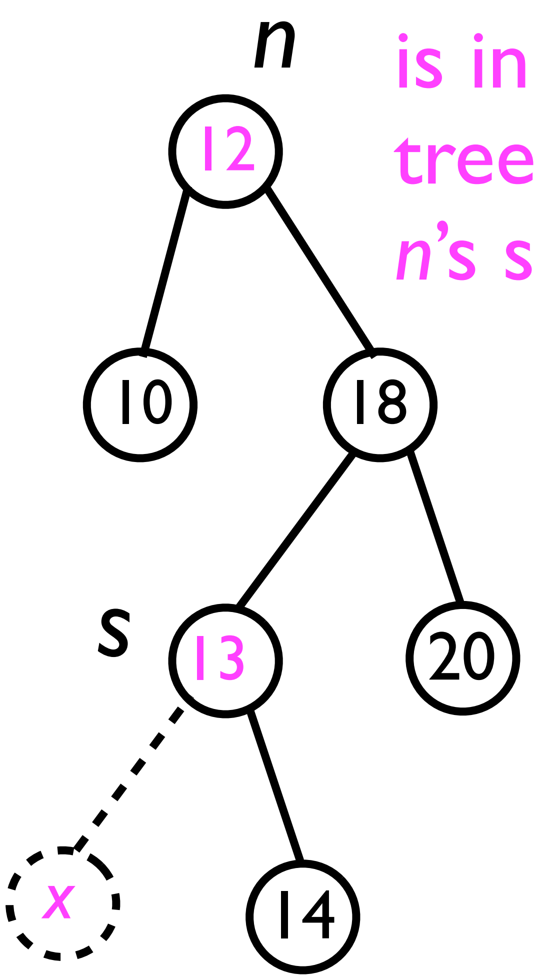
Successor of node with two children

- Example:
    - Let *n* be node 12.
    - Then *n*'s successor *s* is 13.
        - *s* only has one child.



Successor of node with two children

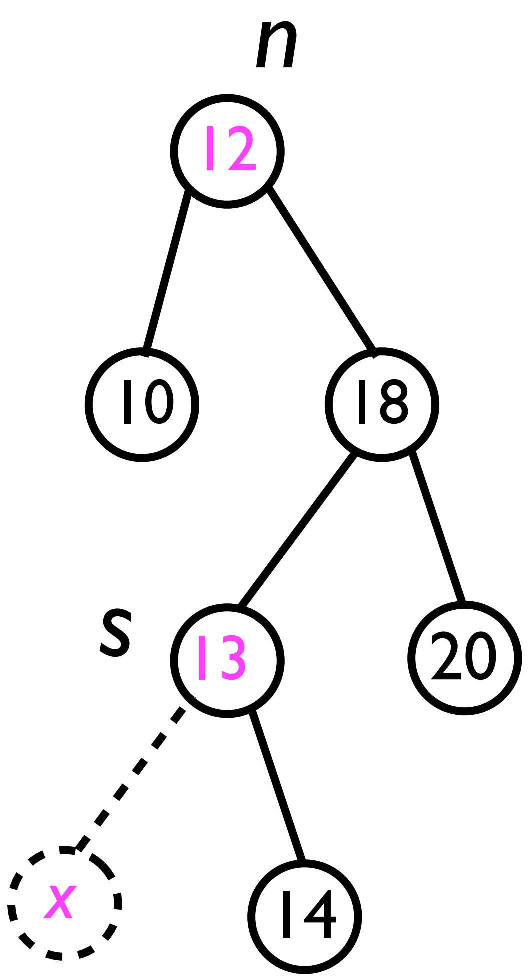
- Example:
  - Let *n* be node 12.
  - Then *n*'s successor *s* is 13.
    - *s* only has one child.
  - Suppose *s* had two children.
    - Then it would have a left child, *x*.
    - *Then x would have to be n's successor.*



Since *x* is still in *n*'s right sub-tree, *x*>12. And since *x* is in *s*'s left sub-tree, *x*<13. So, *x* is *n*'s successor.

Successor of node with two children

- We conclude that, if *n* has two children, then its successor *s* cannot have two children.
- Hence, removing *s* amounts to either just "snipping it off" (case 1), or "slicing around it" (case 2).
- Hence, the **`remove`** method will in fact terminate.



`remove(o)`

- We can finally define the **`remove(o)`** method:

```
void remove (T o) {
  final Node<T> node = findNode(_root, o);
  removeNode(node);
}

void removeNode (Node<T> node) {  // Helper method
  if (node._leftChild == null &&
      node._rightChild == null) {
    // “Snip” node from its parent
  } else if (node._leftChild == null ||
      node._rightChild == null) {
    // “Splice around” node
  } else {
    final Node<T> successor = findSuccessor(_root, o);
    node._data = successor._data;
    removeNode(successor);
  }
}
```

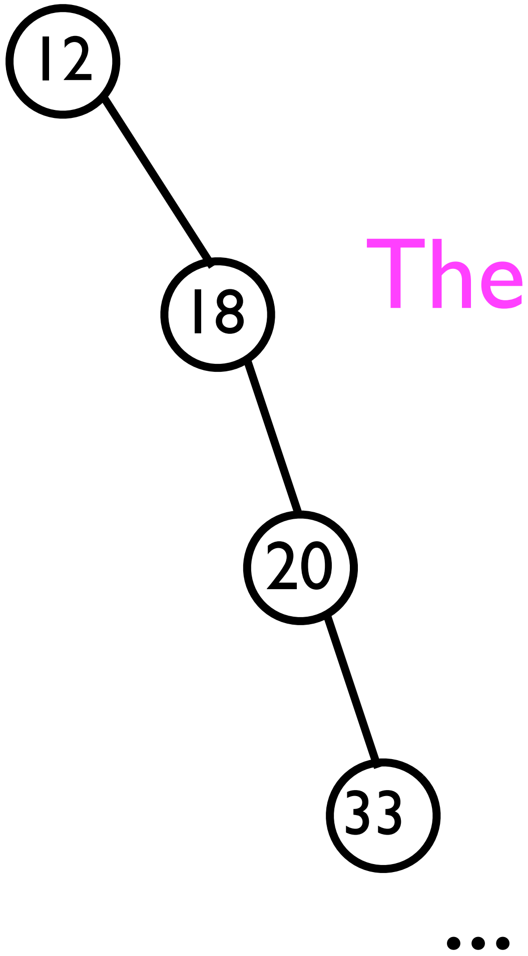
BSTs: Time costs of methods

- All of the fundamental operations -- **`add(o)`**, **`find(o)`**, **`remove(o)`**, and **`findLargest`**/**`findSmallest`** -- take time *O*(*h*), where *h* is the height of the BST.
- In the *average case*, the height *h* of the BST is log *n*.
- What about in the *worst case*?

BSTs:
Time costs of methods

- In the *worst case*, the user will call **`add`** and **`remove`** in an "unfortunate" order, resulting in a "degenerate" BST of the following variety:

- In this case, the height of the BST is *n* -- and hence the fundamental BST operations would also be *O*(*n*).



The "BST" is just a linked list!

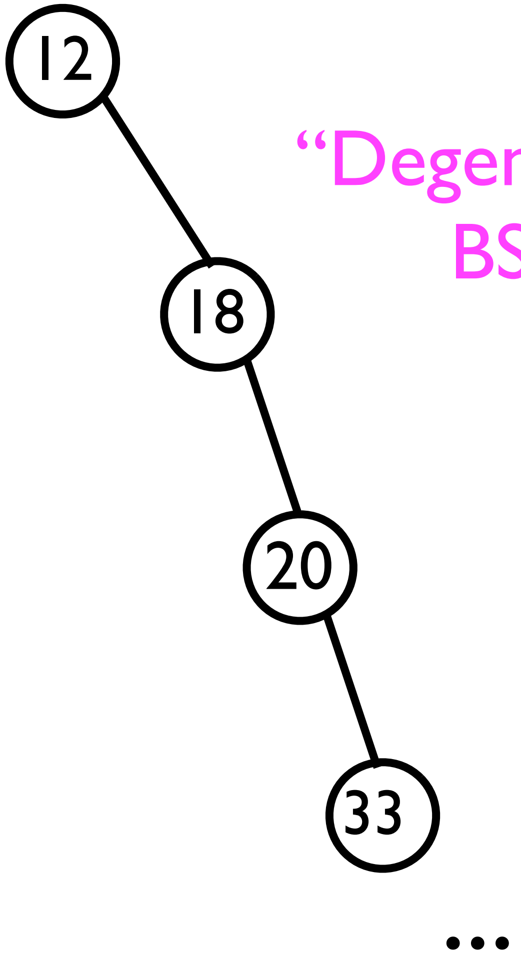
Balancing BSTs

- To prevent this “worst-case” condition from occurring, we need to employ some form of “tree balancing” to keep the tree from degenerating into a linked list.
- Two prominent data structures which ensure a *balanced tree* include:
  - AVL trees.
  - Red-black trees.

AVL trees.

Maintaining balance

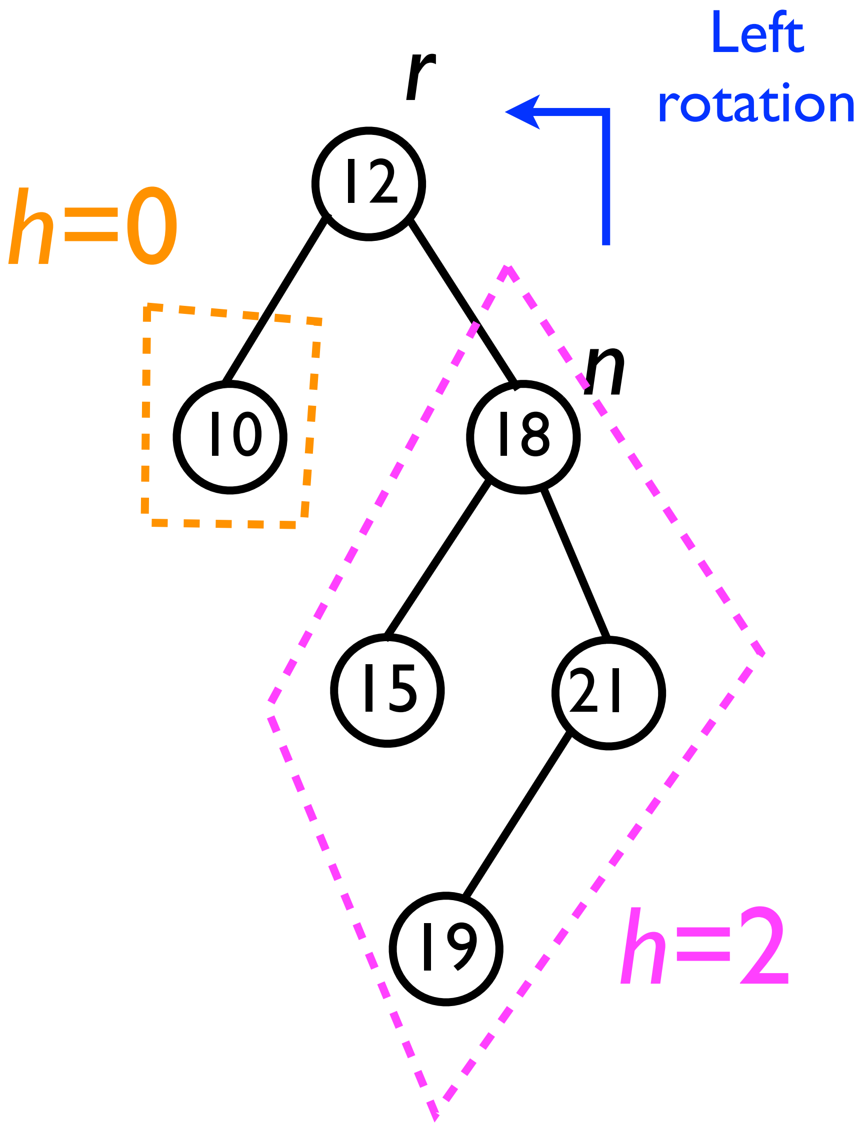
- The time cost of the fundamental add/find/remove operations in BSTs depends on the *height* of the BST.
- Given an “unfortunate” sequence of add/remove operations, the BST can “degenerate” into a long “chain” of nodes of height *n*.
  - Hence, in the worst case, the time cost of the fundamental BST operations is $O(n)$.
  - It would be beneficial to *prevent* this worst case from ever occurring.



“Degenerate” BST.

Maintaining balance

- Fortunately, it turns out that BSTs can be "fixed" to store the *same elements*, but to have a *smaller height*.
- Consider the BST on the right (with root *r*) with height 3.
  - It is *unbalanced* -- height of left sub-tree is 0, height of right sub-tree is 2.
- We can "fix" this BST to have *equal* height on both sub-trees by "rotating" node *n* towards *r*.



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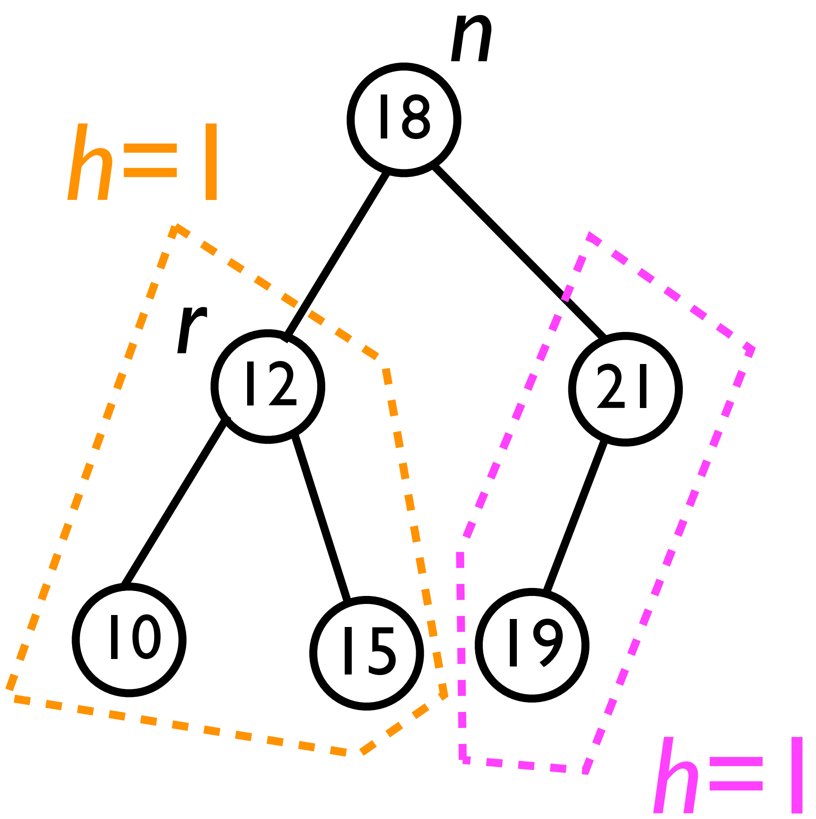


New root is *n*.
Height of BST is 2.
Left and right sub-trees both have height 1 (the BST is *balanced*).

Maintaining balance

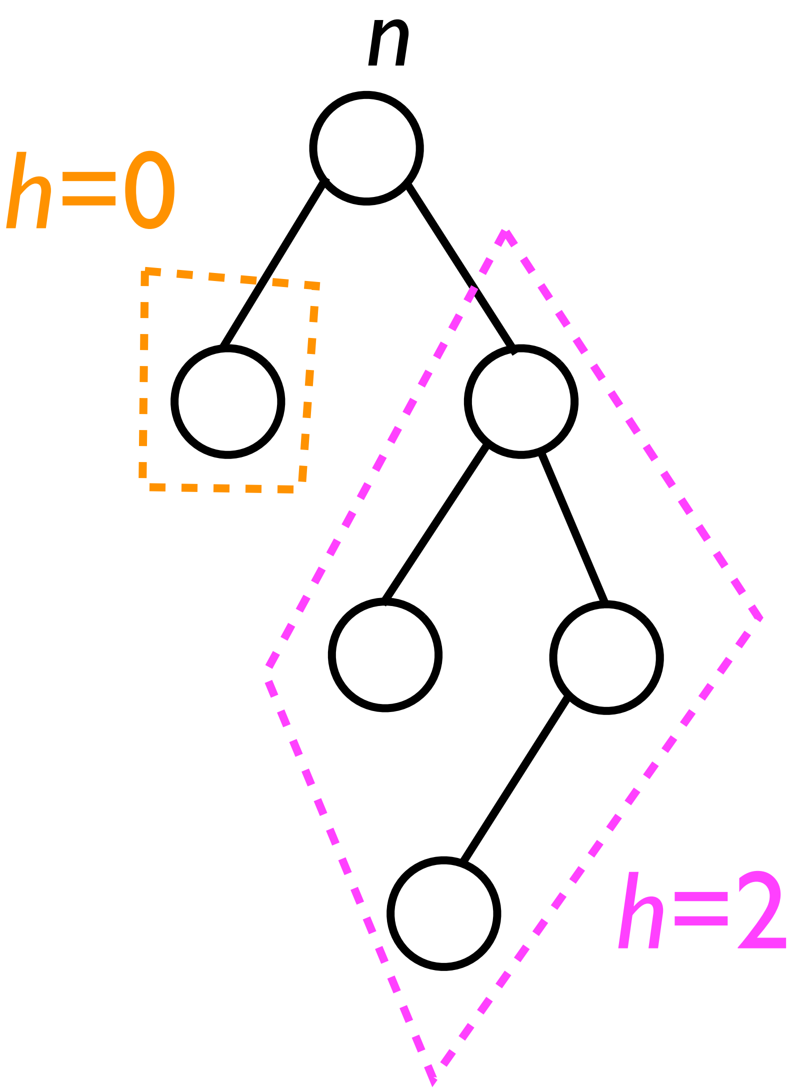
- By rotating nodes to either "up-to-the-left" or "up-to-the-right", we can restore *balance* to a BST and thereby *decrease its height*.
- The rotations will take place whenever the user `adds` or `removes` a node from the BST.
- By rotating properly, we can ensure that the BST remains balanced or "almost balanced" at all times.
- This system of node rotations was first developed in 1962 by G.M. Adelson-Velskii and E.M. Landis; hence, we call this technique an AVL-tree.

AVL trees

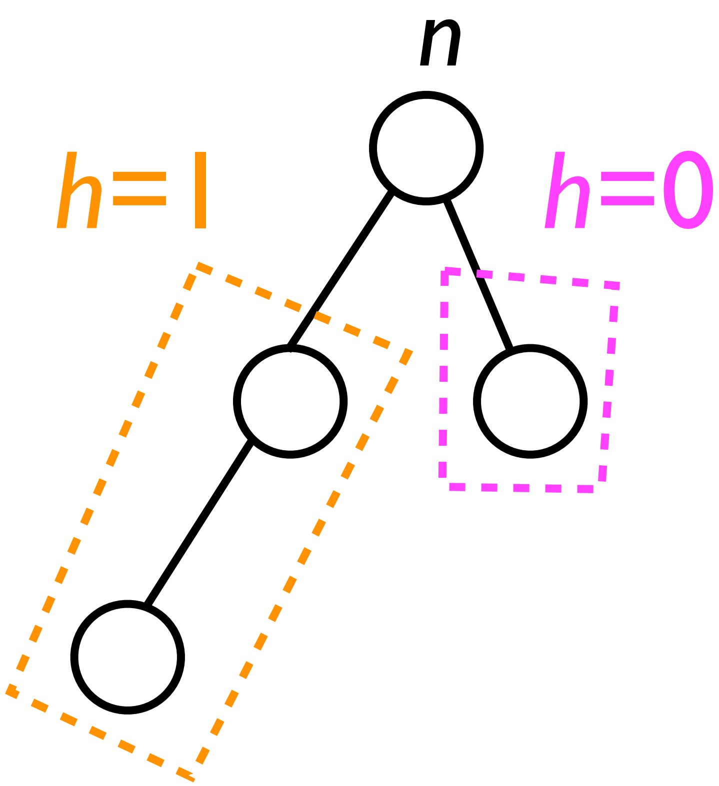
- An AVL tree is a BST in which two kinds of rotations -- *left-rotations* and *right-rotations* -- are applied to nodes as necessary, in order to keep the *balance* of each sub-tree within certain limits.
- The *balance* of a node *n* is the *difference in height* between *n*'s left sub-tree minus its right sub-tree.
  - A non-existent sub-tree is defined to have height 0.
- Rotations are applied to nodes during the **`add`** and **`remove`** methods to keep *every* node's balance within -1 and +1 (inclusive).

Height and balance

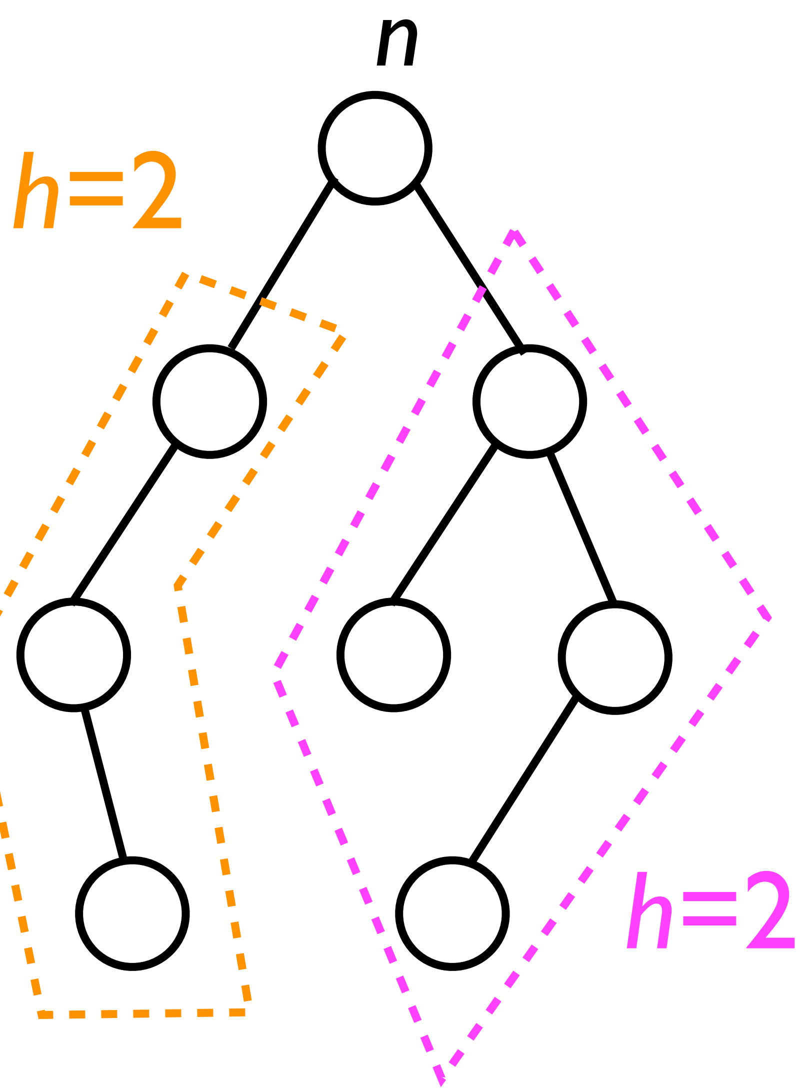
Balance = -2



Balance = +1



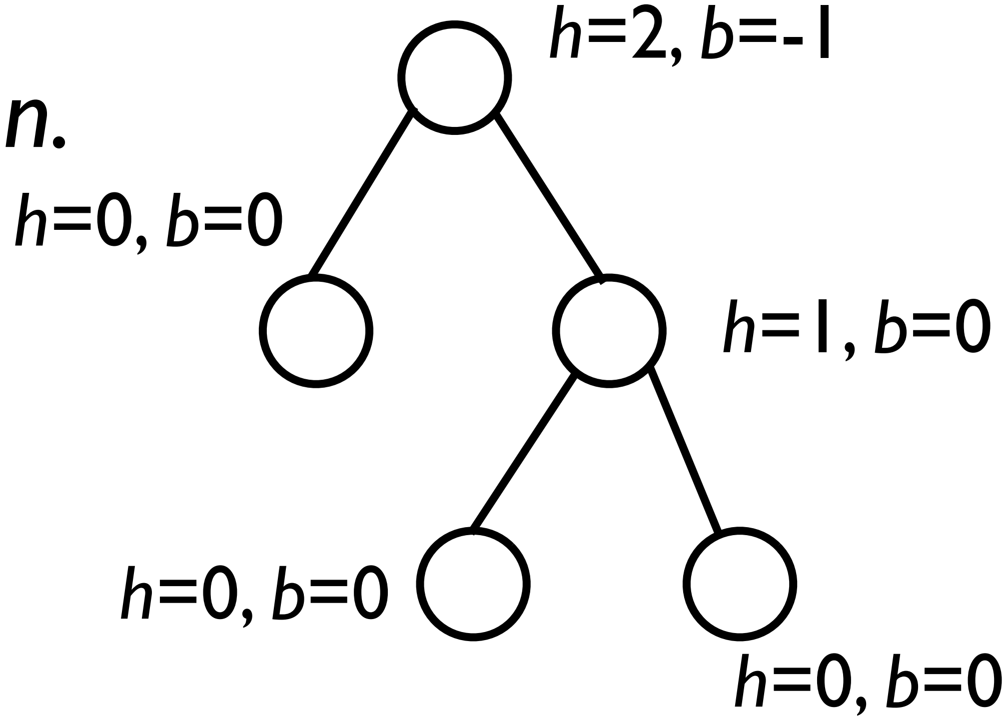
Balance = 0



Height and balance

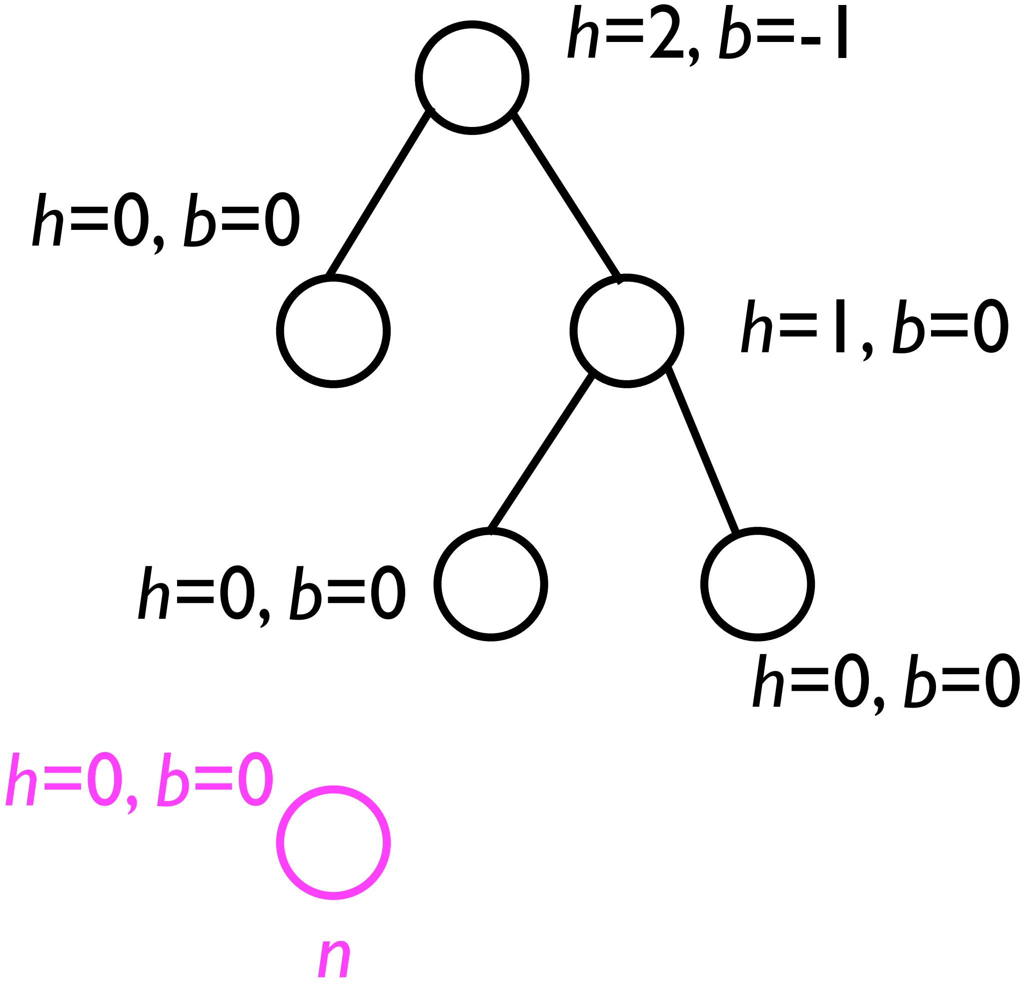
- AVL trees require that each node *n* record its *balance* as well as the *height* of the sub-tree rooted at *n*.
  - We can store these as extra instance variables in the **`Node`** class:

```
class Node<T> {
  Node<T> _parent;
  Node<T> _leftChild, _rightChild;
  int _balance, _height;
}
```



Adding a new node

- Whenever we add a new node *n*, we set its `_height` and `_balance` both to 0.
- We attach *n* as a left/right child of its parent.
- We must then recursively update the height and balance of all nodes from *n* up through the root of the whole BST.



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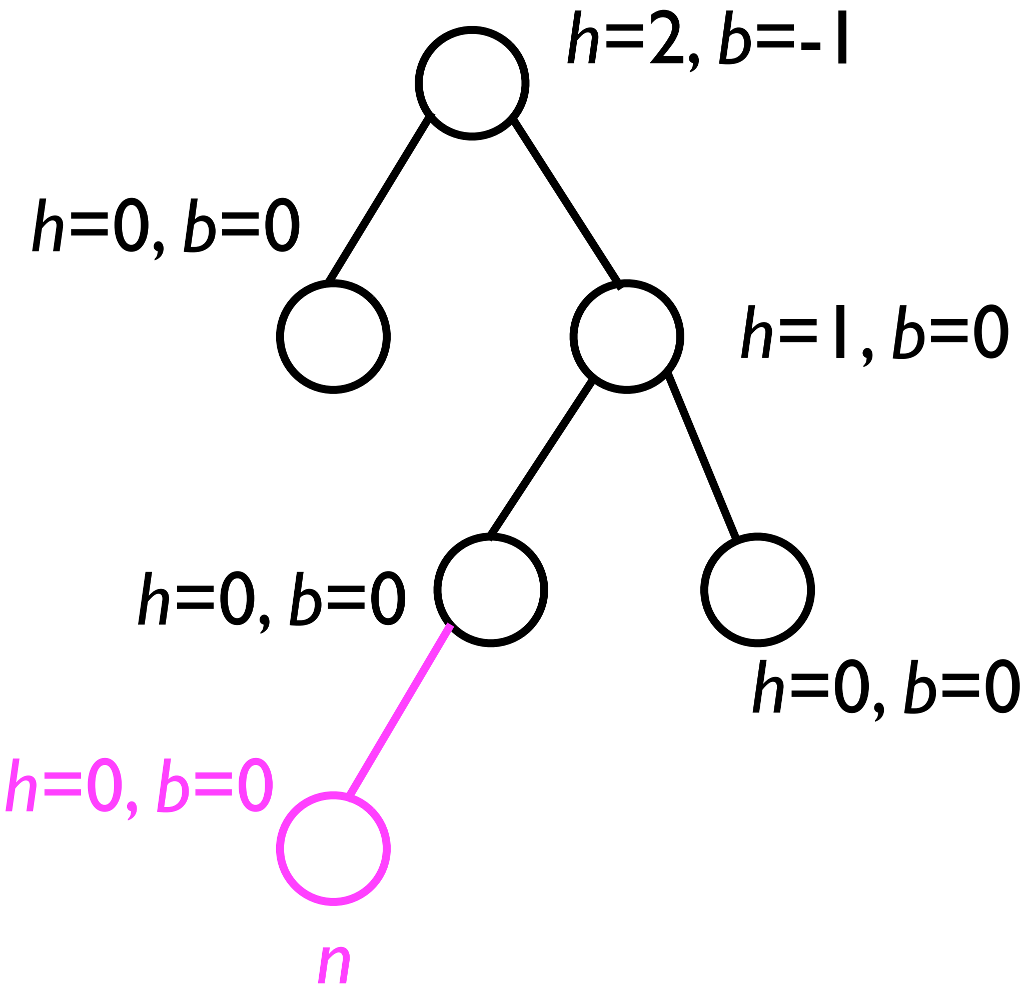

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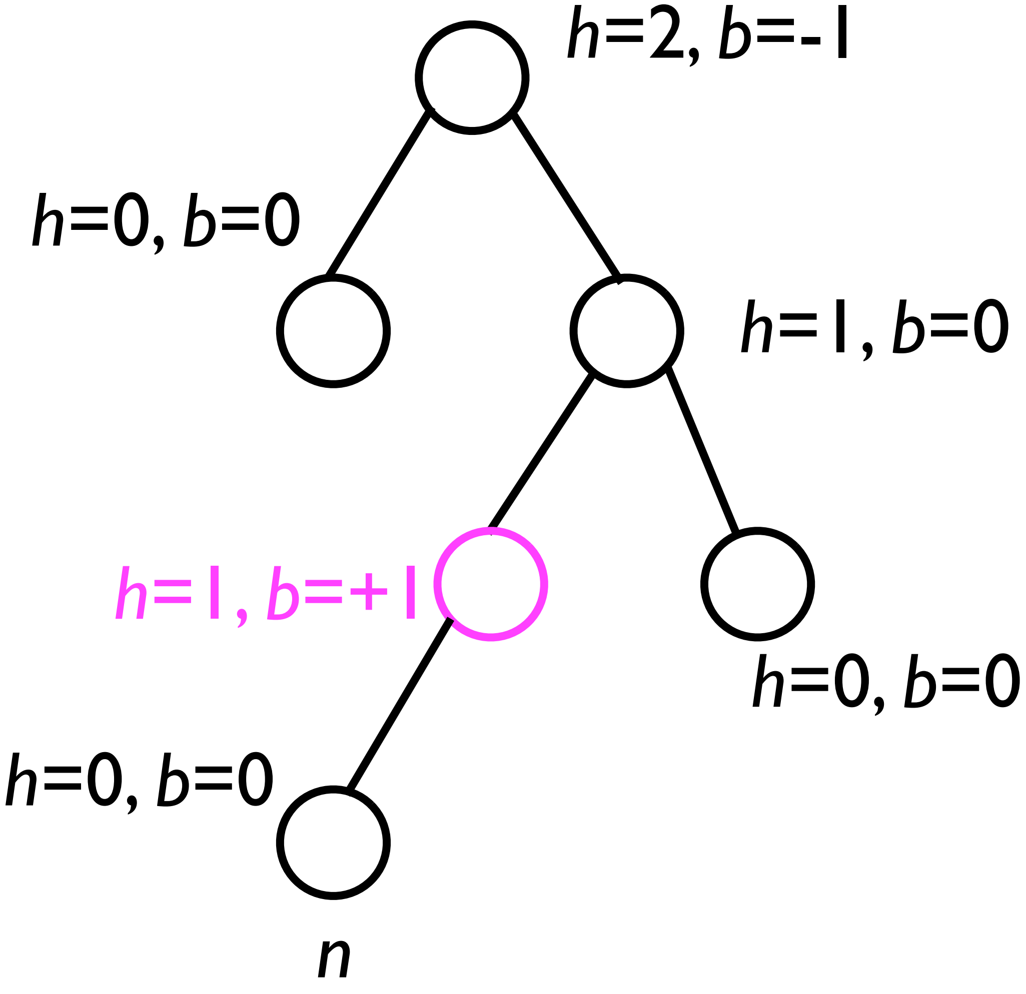

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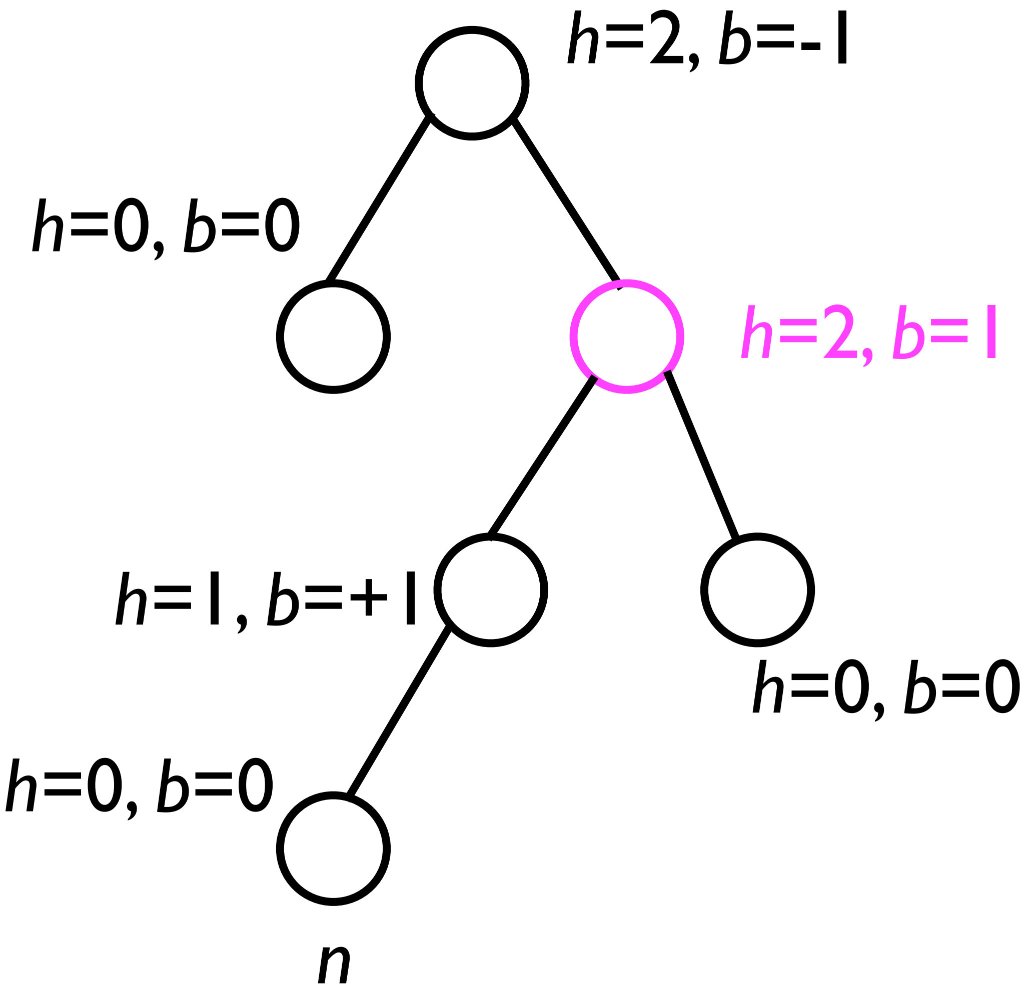

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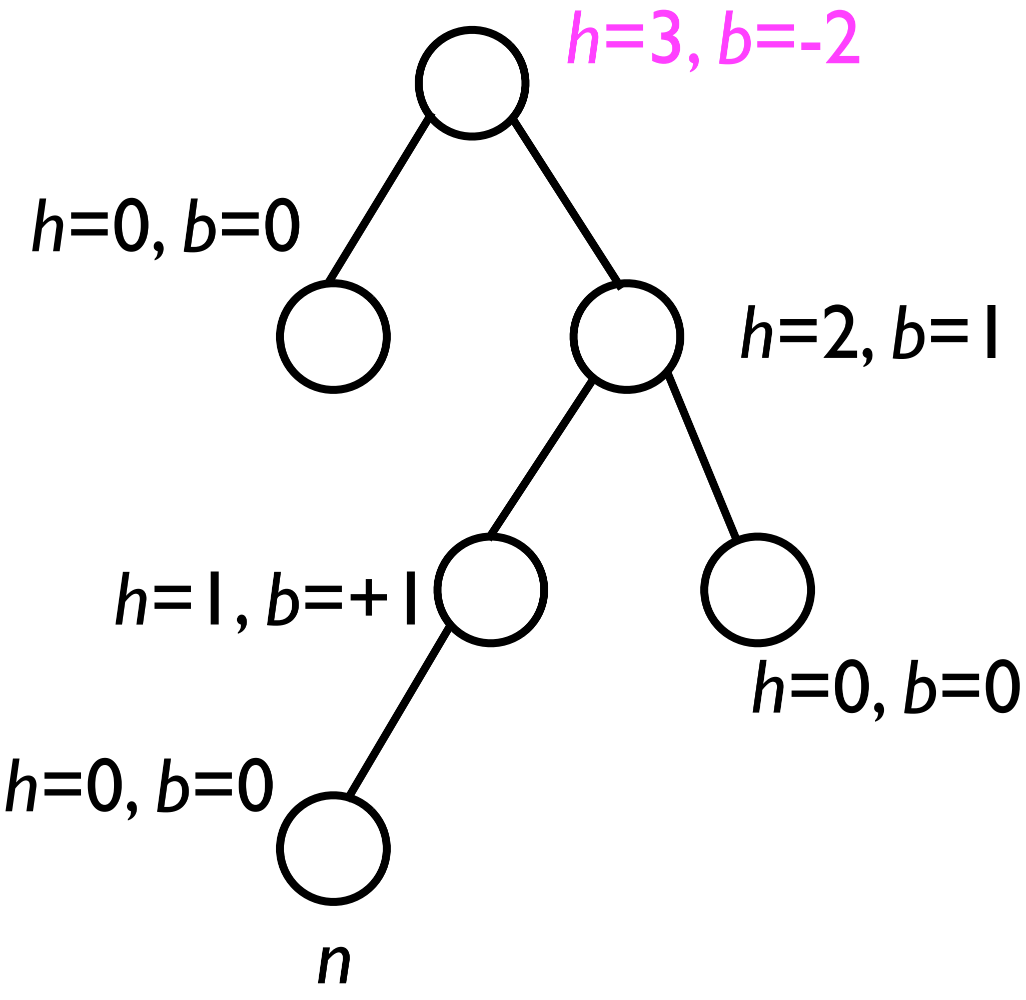


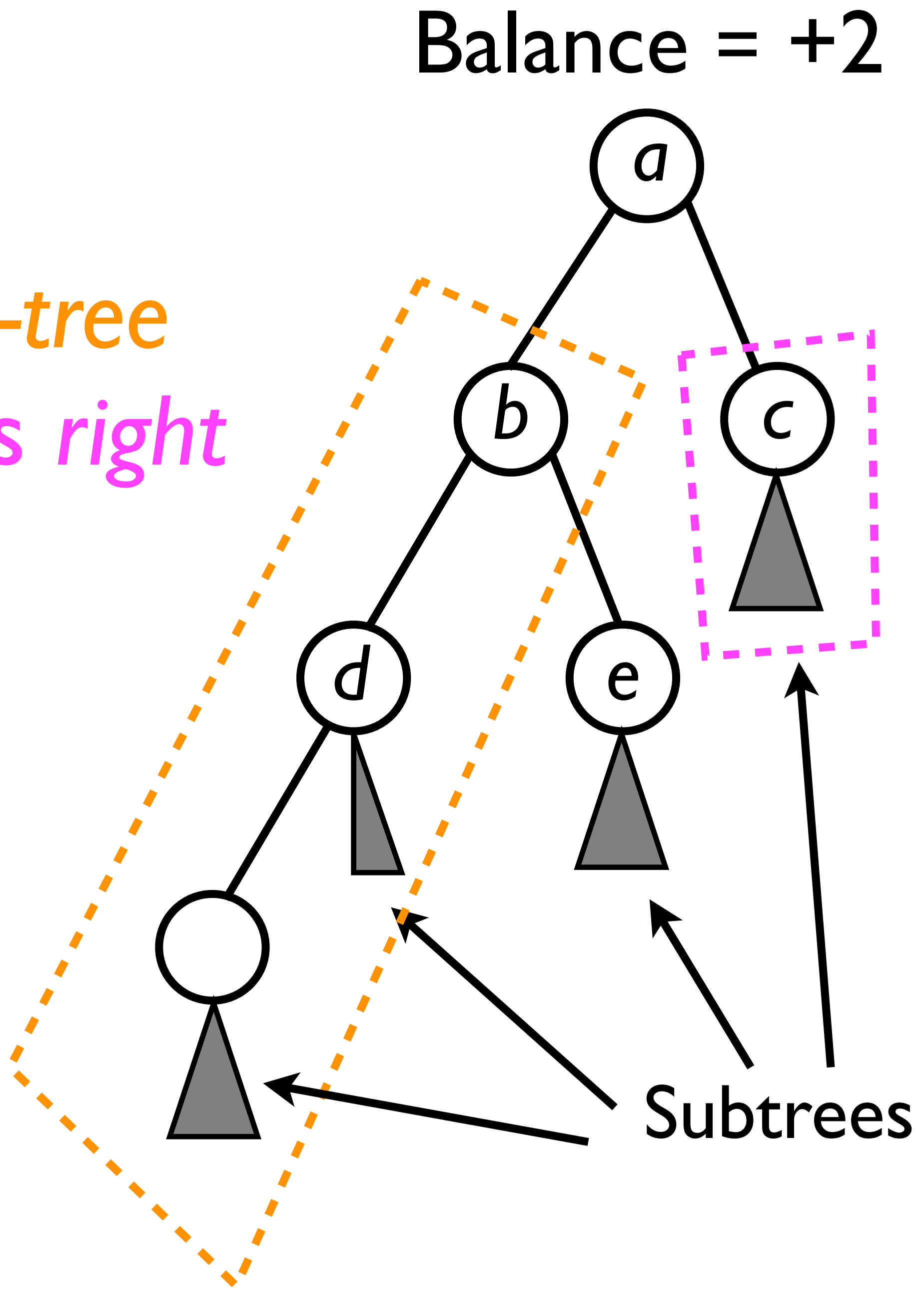
Correcting imbalances

- Suppose, when recursively updating the height and balance data, we determine that the balance of a node *n* is either -2 or +2.
    - *n* is considered *imbalanced*.
- Then we must apply an AVL rotation to *correct the imbalance*.
- Different rotations apply to different node configurations...

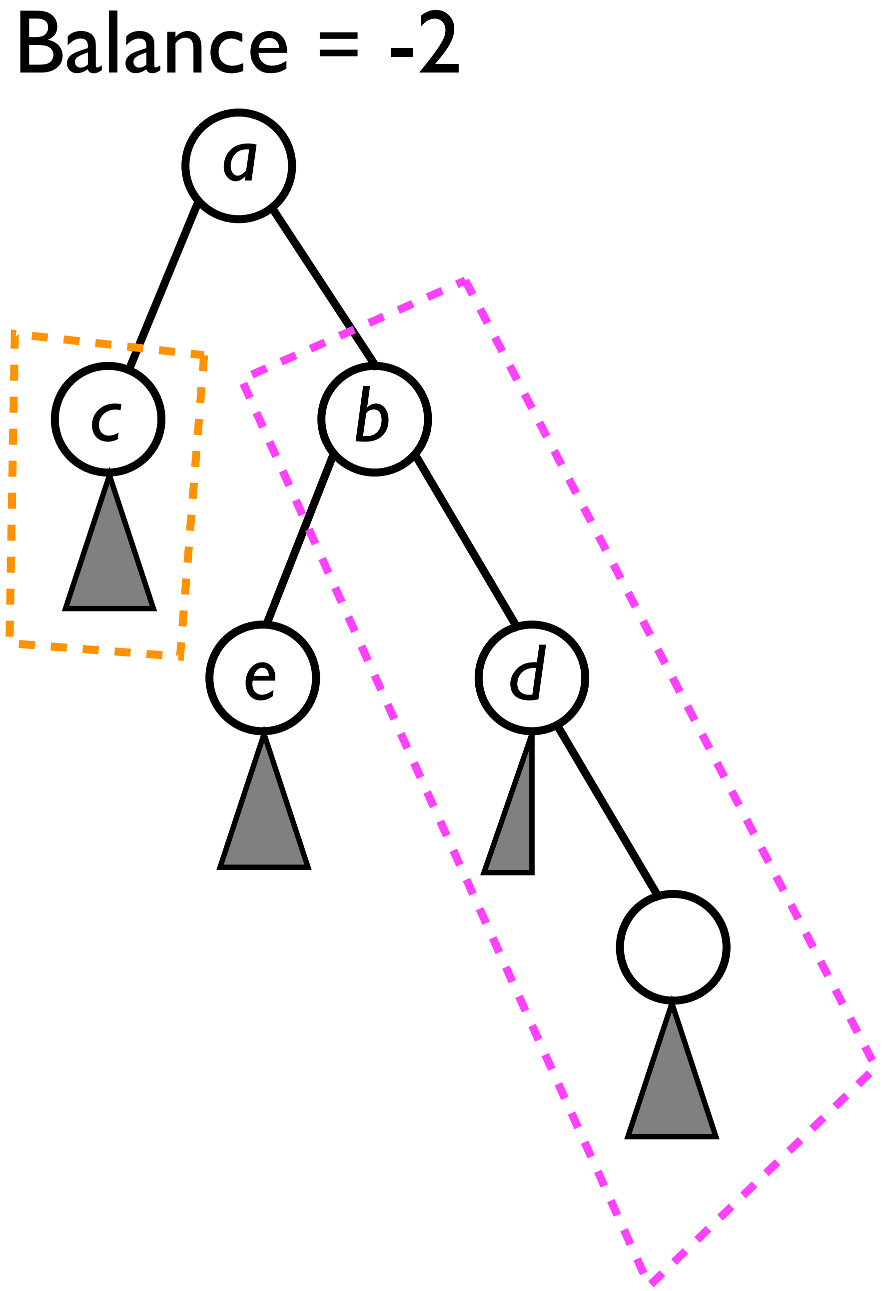
Imbalanced node configurations

The *Left child's Left sub-tree of a* is 2 higher than *a's right sub-tree*.

This case is called LL.



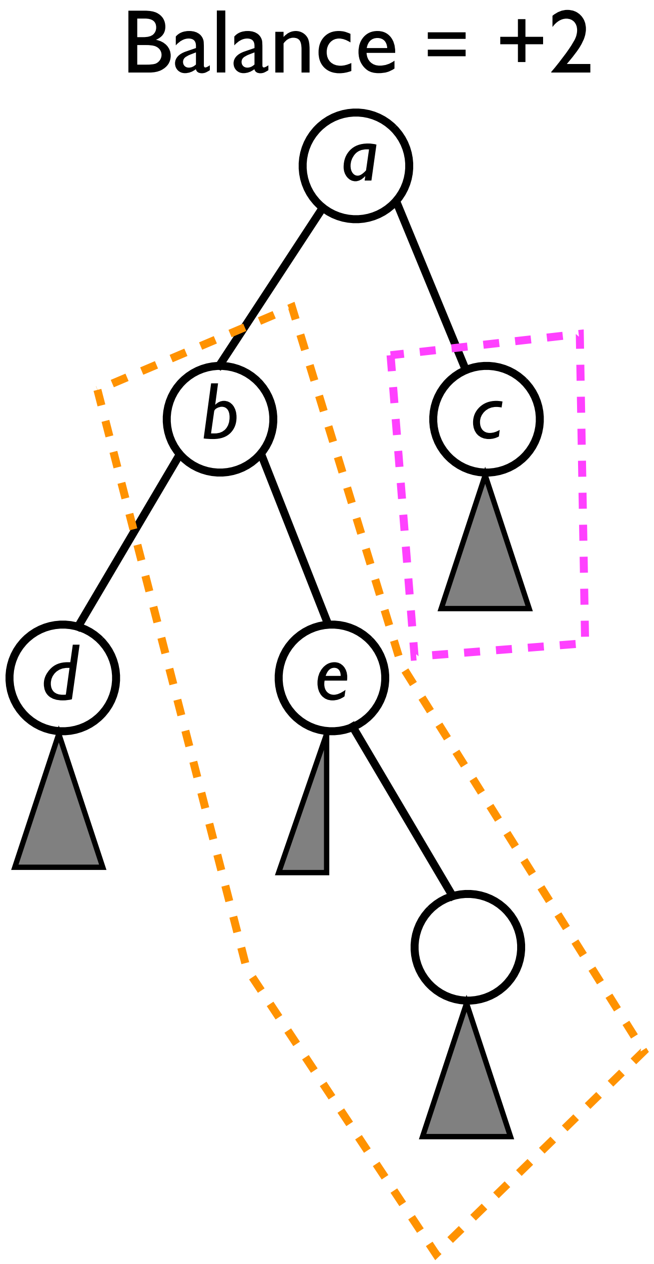
Imbalanced node configurations



The *Right child's Right sub-tree of a* is 2 higher than *a's left sub-tree.*

This case is called RR.

Imbalanced node configurations



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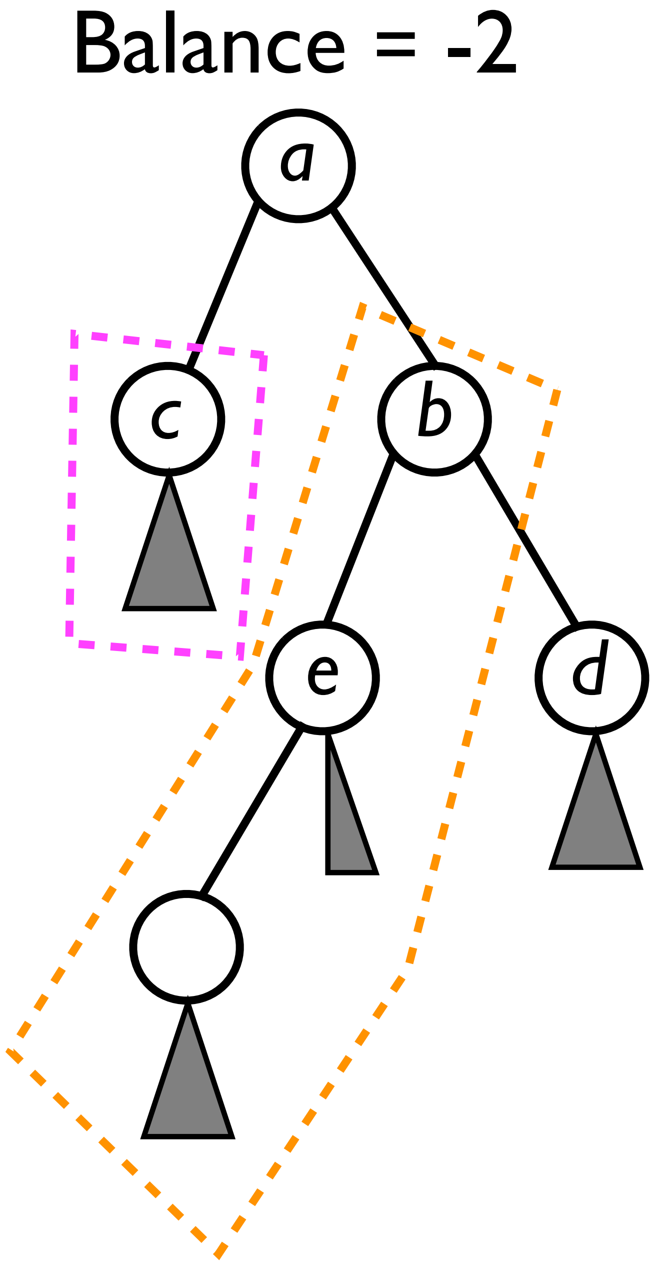
Imbalanced node configurations

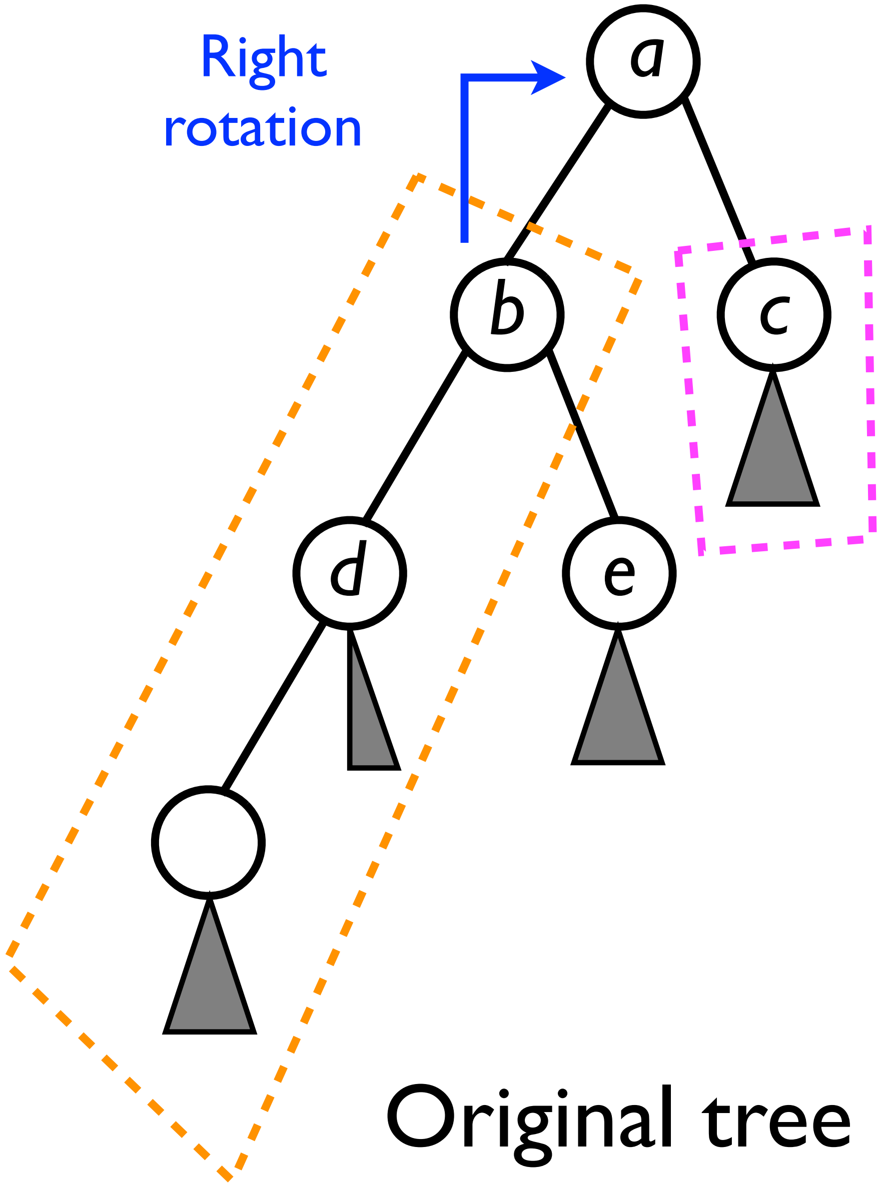


The *Right child's Left sub-tree of a* is 2 higher than *a's left sub-tree*.

This case is called RL.

Fixing configuration LL

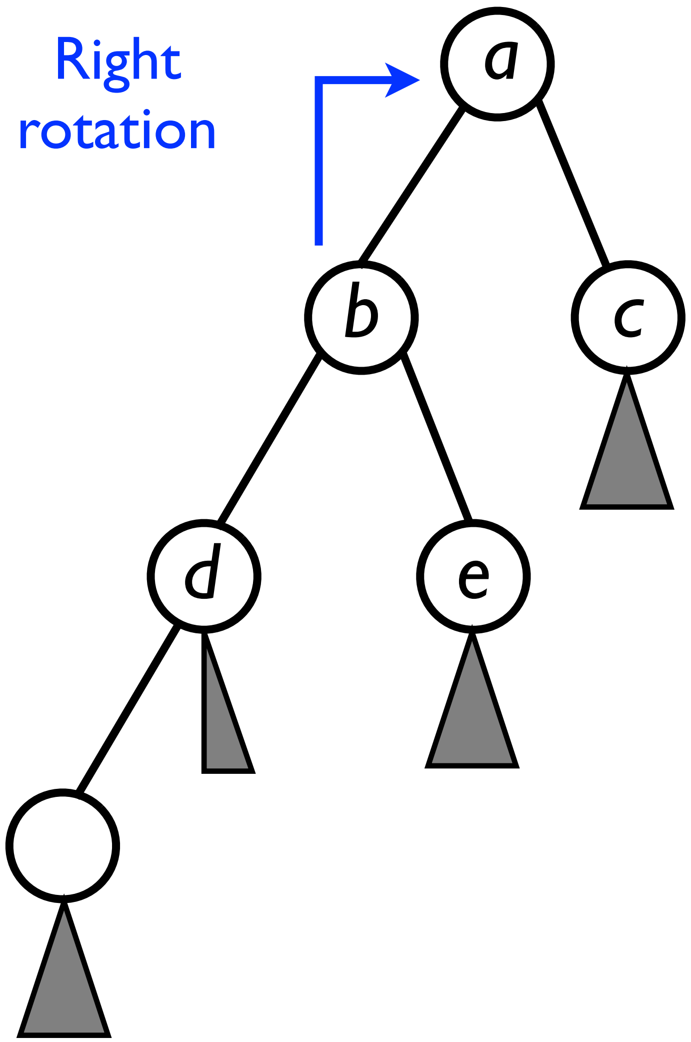
- To fix the imbalance in node *a*, we will perform a *right rotation* of node *b* towards *a*.



Original tree

Fixing configuration LL

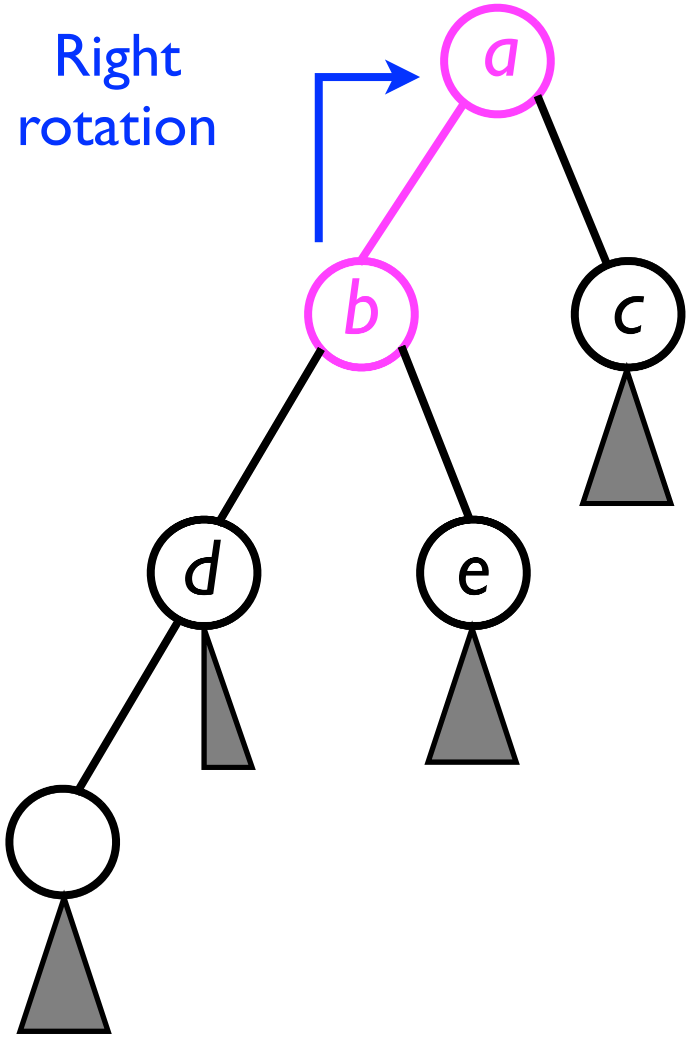
- To fix the imbalance in node *a*, we will perform a *right rotation* of node *b* towards *a*.



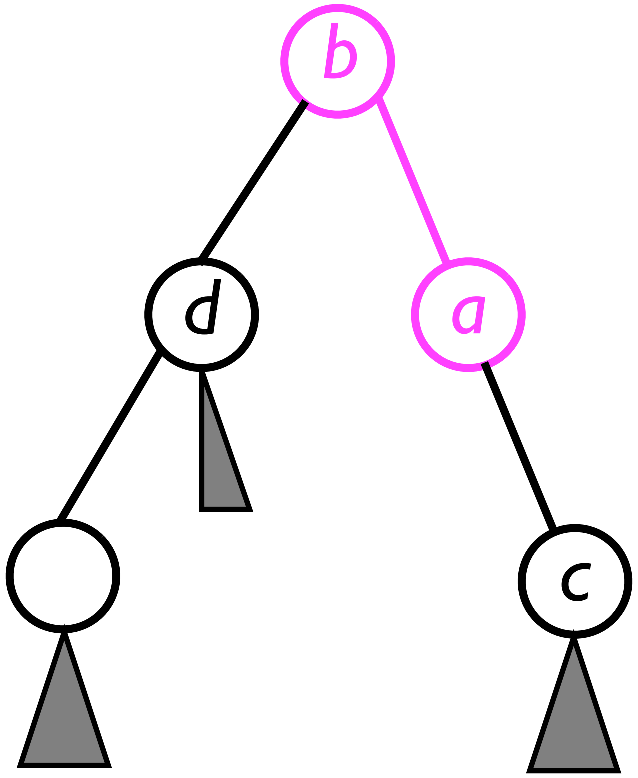
Original tree

Fixing configuration LL

- To fix the imbalance in node *a*, we will perform a *right rotation* of node *b* towards *a*.



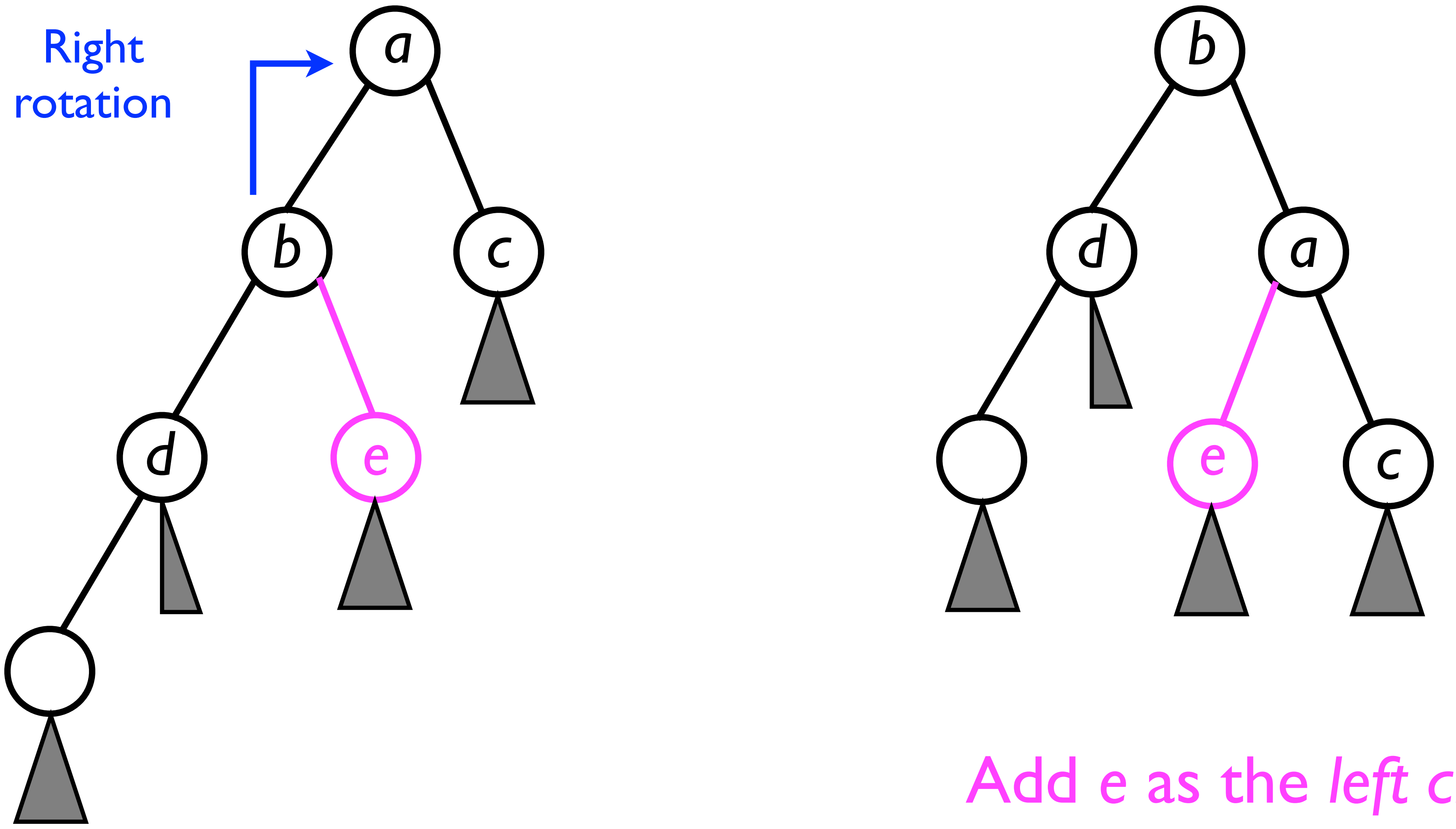
Original tree



Make *a* the *right child* of b, and make *b* the new root of the sub-tree.

Fixing configuration LL

- To fix the imbalance in node *a*, we will perform a *right rotation* of node *b* towards *a*.

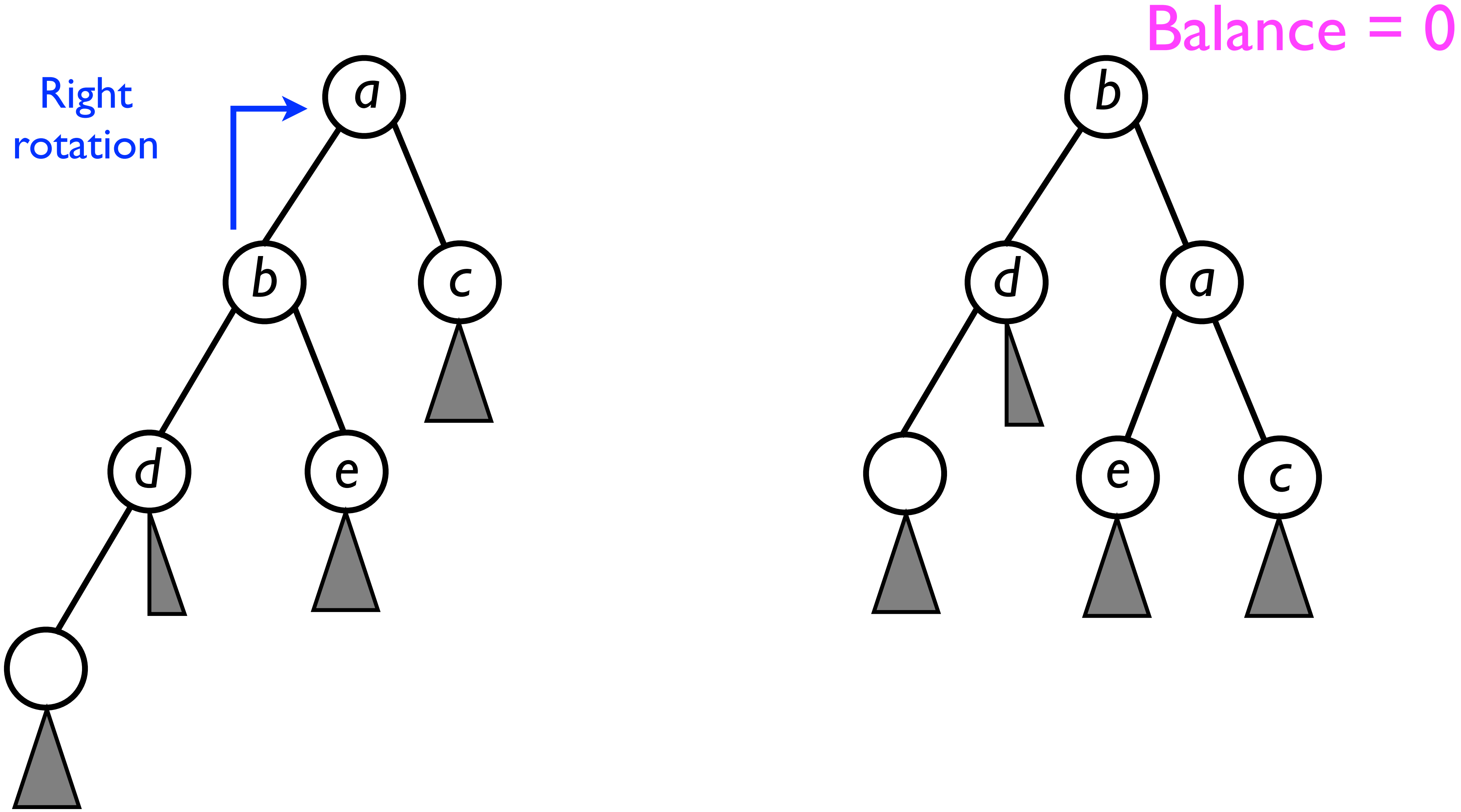


Original tree

Add e as the *left child* of *a*.

Fixing configuration LL

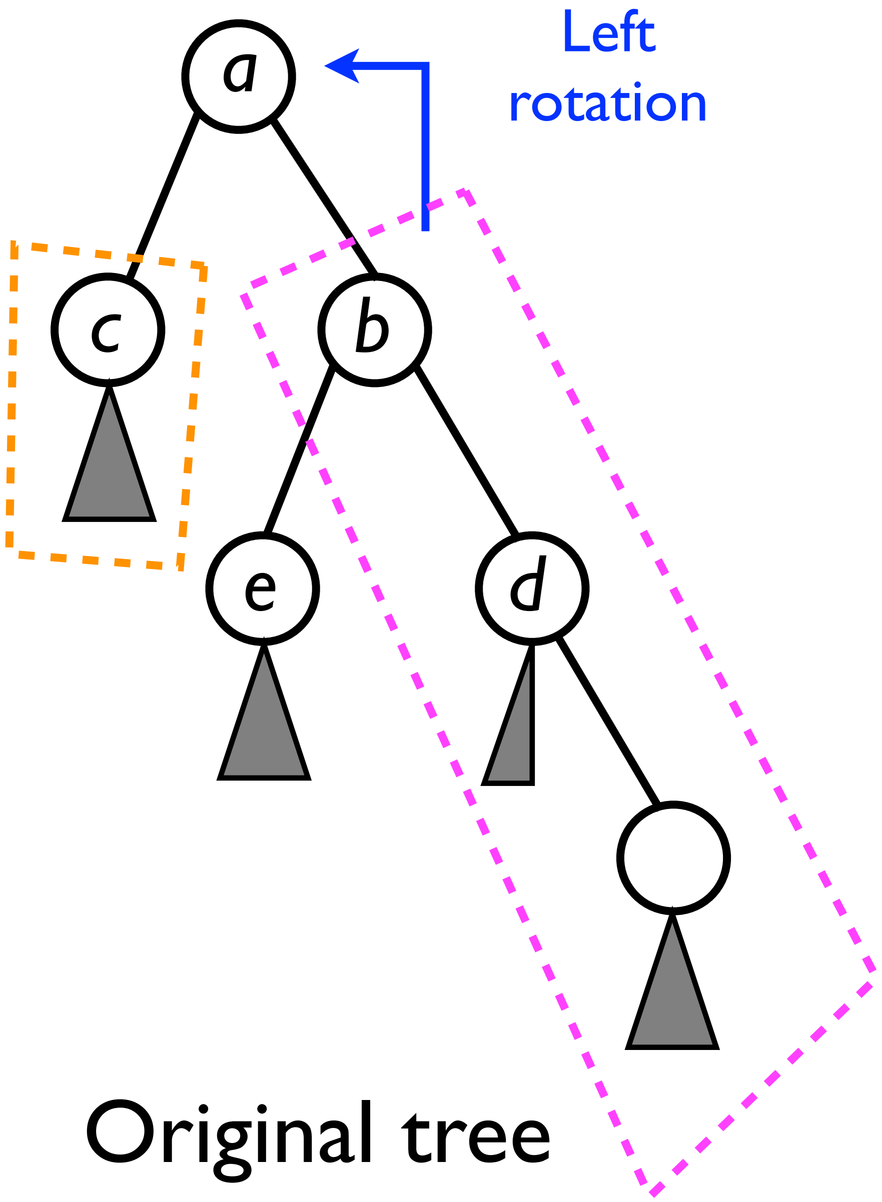
- To fix the imbalance in node *a*, we will perform a *right rotation* of node *b* towards *a*.



Original tree

Fixing configuration RR

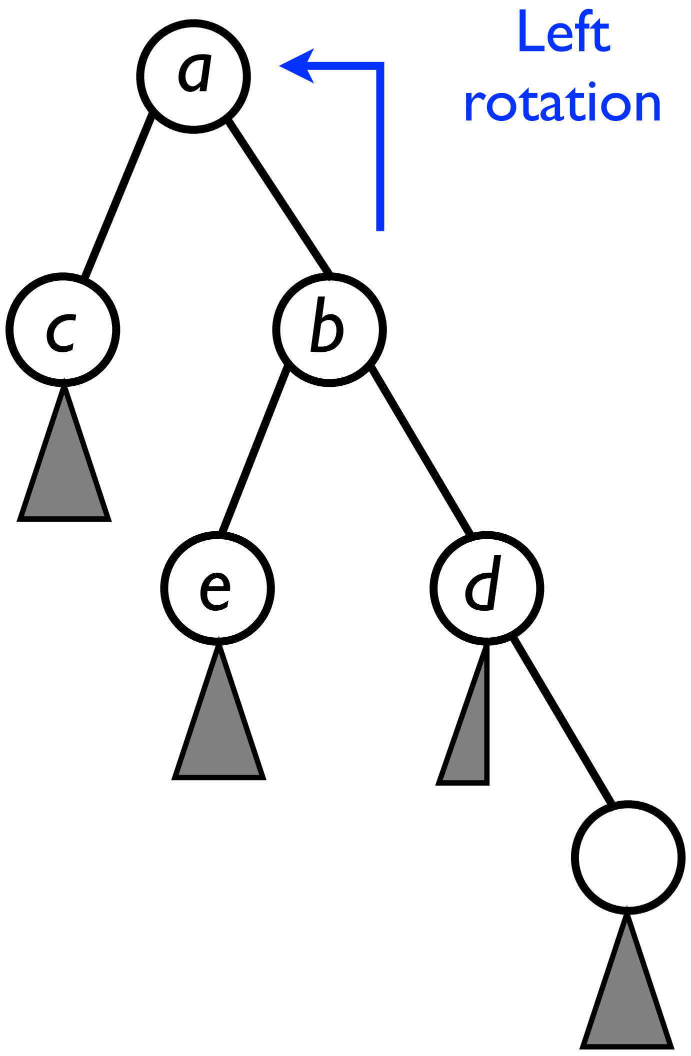
- To fix the imbalance in node *a*, we will perform a *left rotation* of node *b* towards *a*.



Original tree

Fixing configuration RR

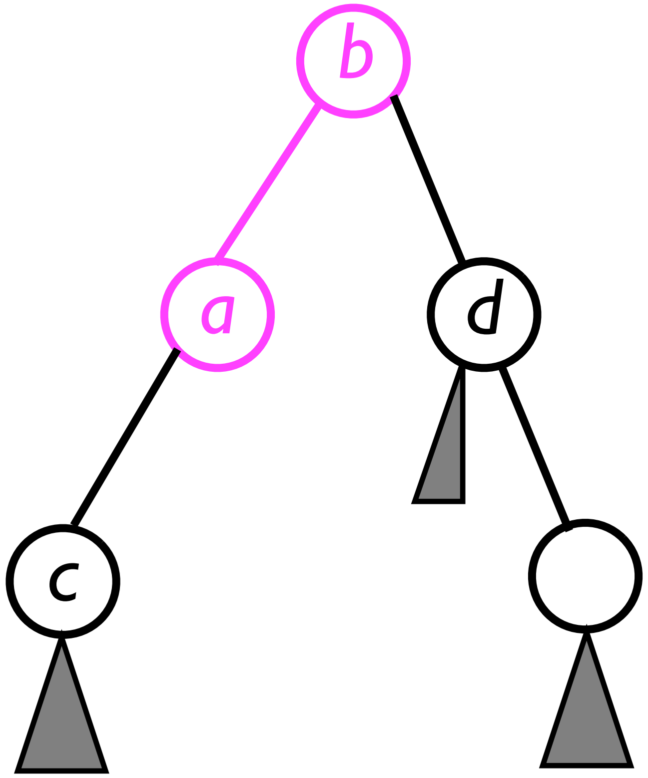
- To fix the imbalance in node *a*, we will perform a *left rotation* of node *b* towards *a*.



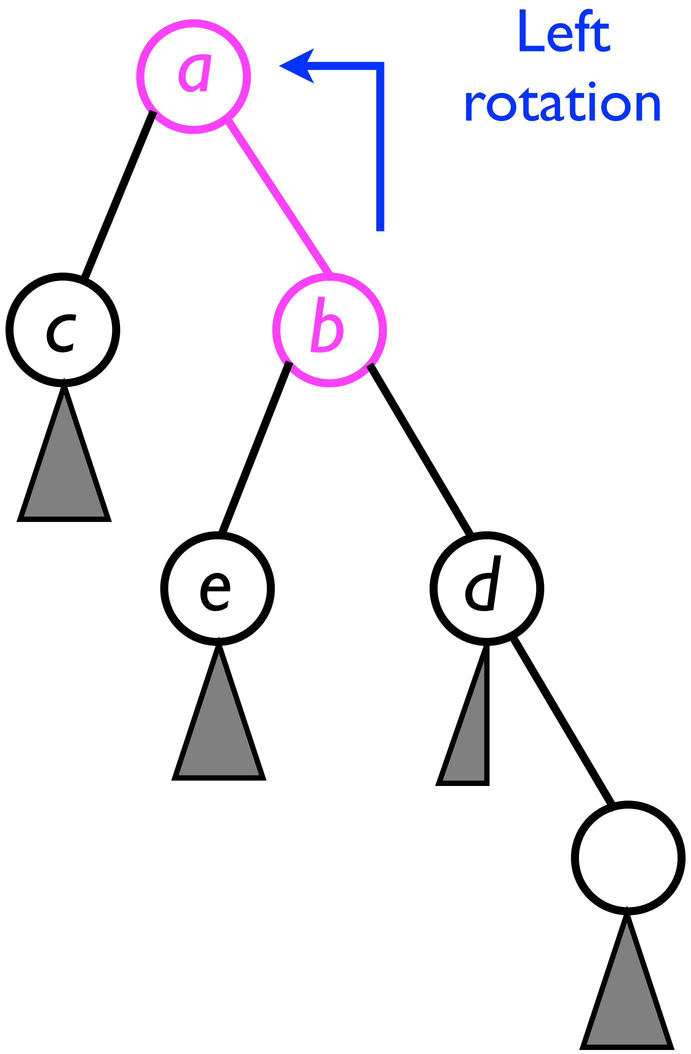
Original tree

Fixing configuration RR

- To fix the imbalance in node *a*, we will perform a *left rotation* of node *b* towards *a*.



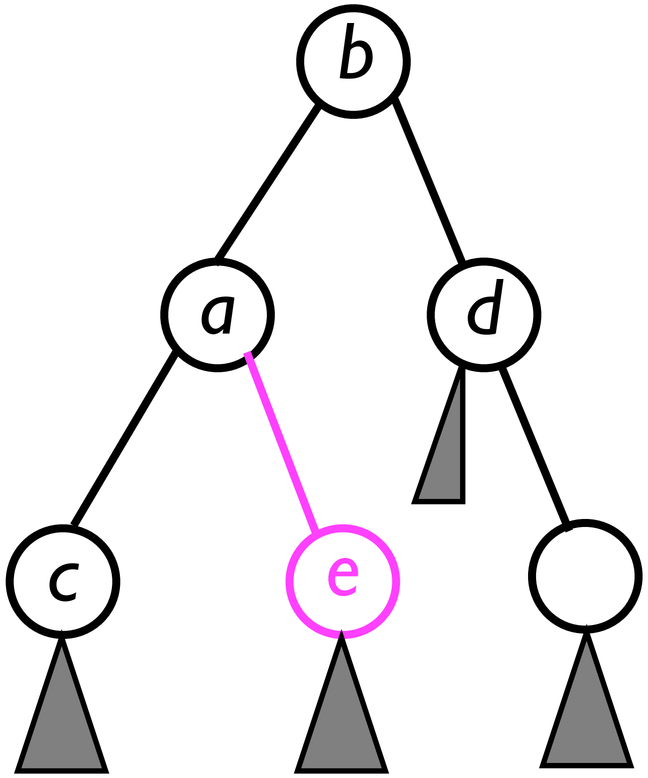
Make *a* the *right child* of b, and make *b* the new root of the sub-tree.



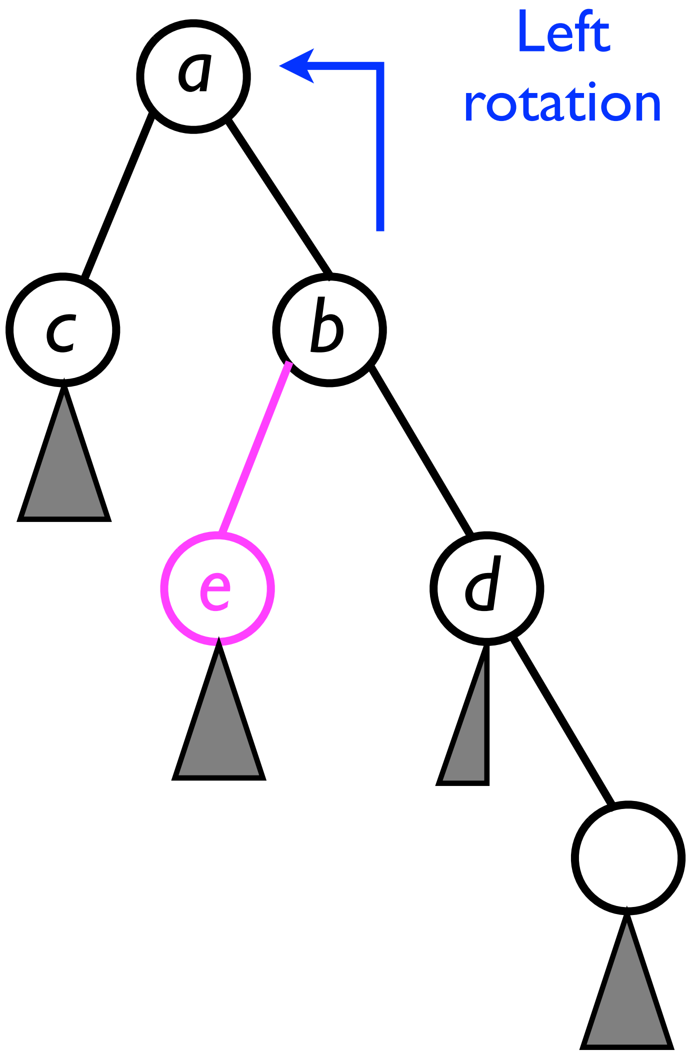
Original tree

Fixing configuration RR

- To fix the imbalance in node *a*, we will perform a *left rotation* of node *b* towards *a*.



Add e as the *left child of a*.

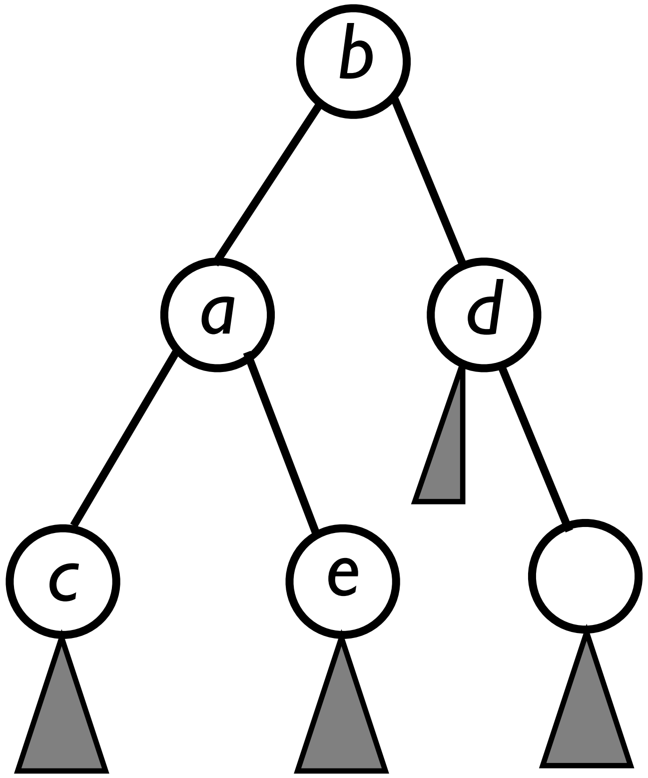


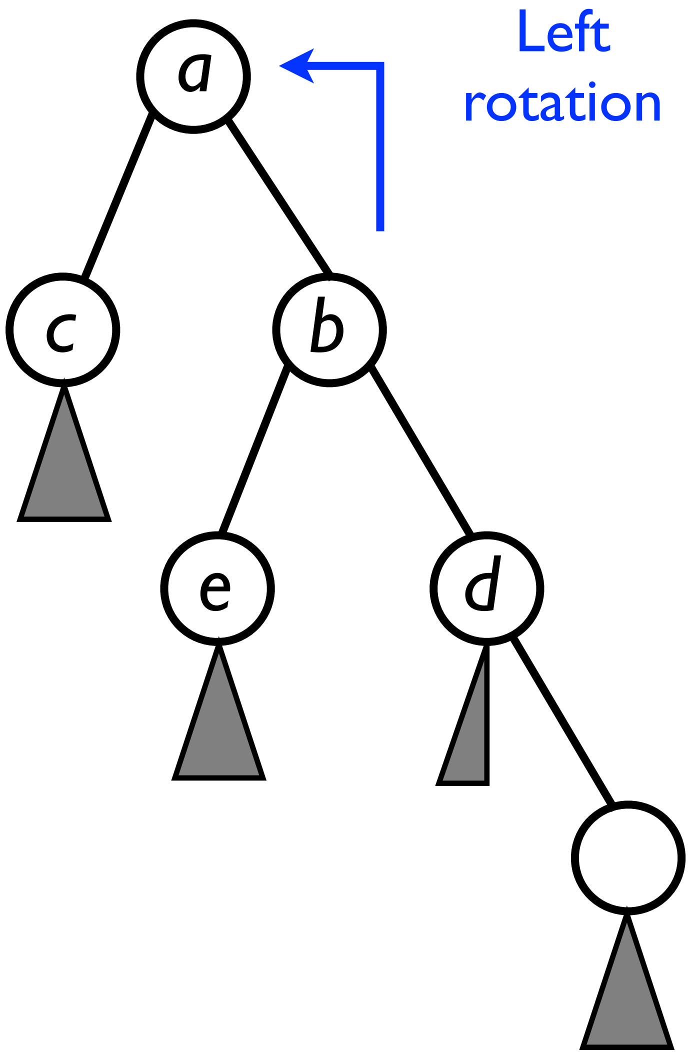
Original tree

Fixing configuration RR

- To fix the imbalance in node *a*, we will perform a *left rotation* of node *b* towards *a*.

Balance = 0





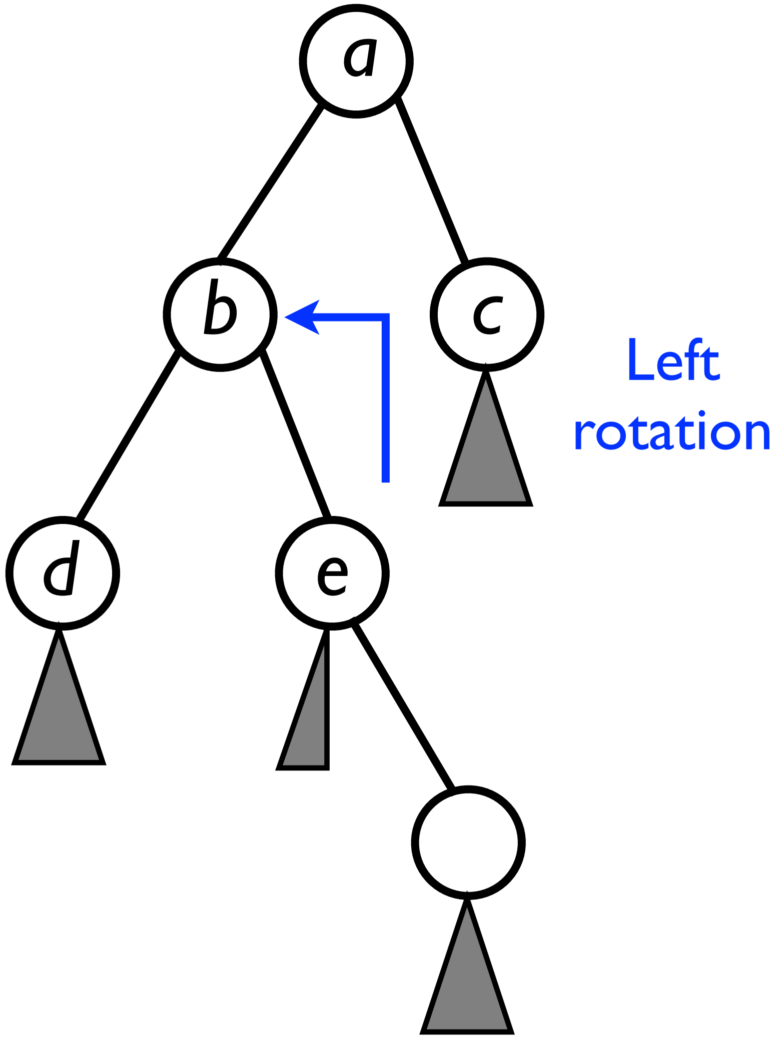
Original tree

Imbalanced node configurations

- Note how LL and RR, as well as LR and RL, are *symmetric* to each other.
  - LL is fixed by *right rotating a*.
  - RR is fixed by *left rotating a*.
- The other two cases -- LR and RL -- can be fixed by *two rotations in succession*.

Fixing configuration LR

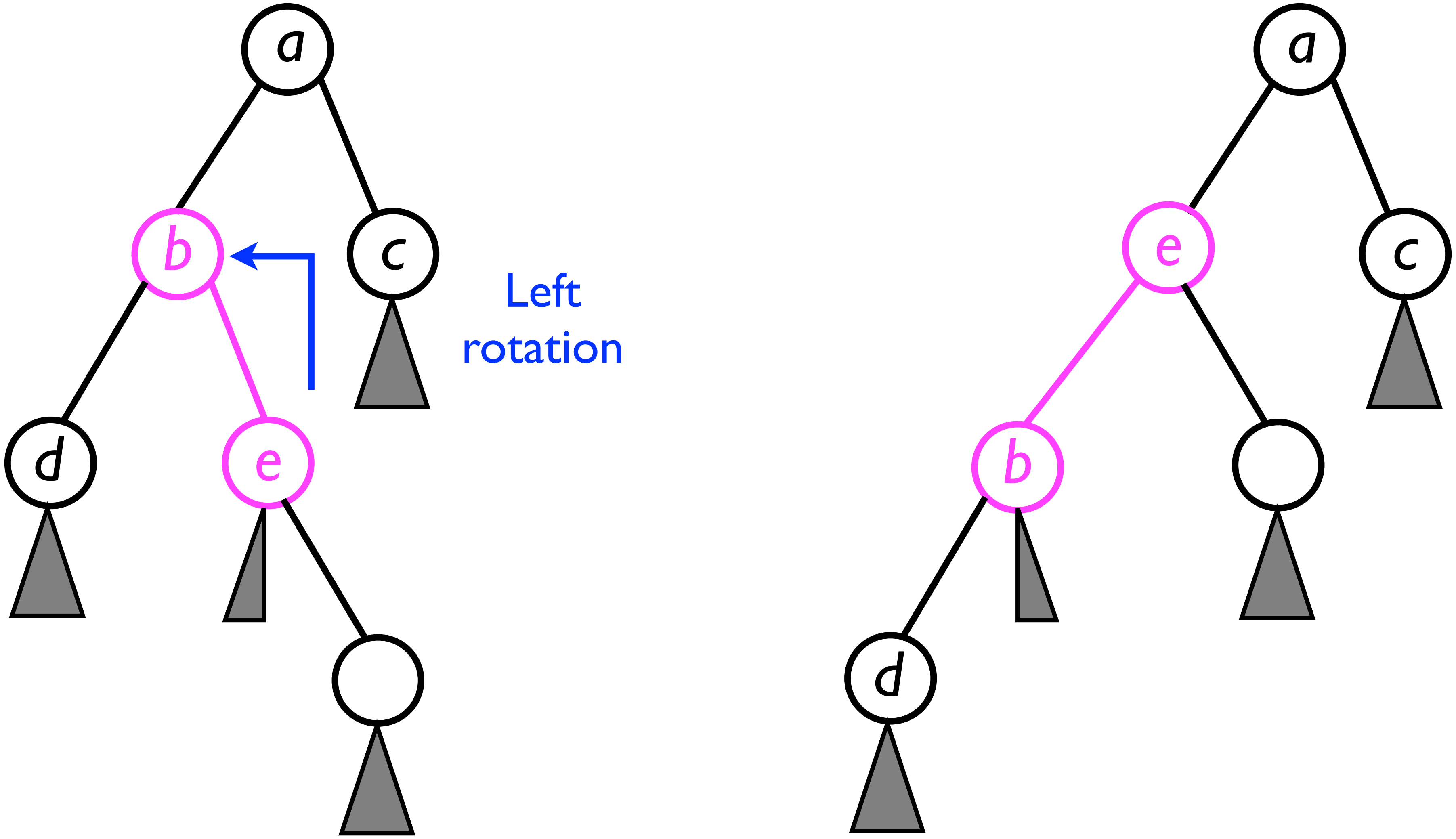
- To fix the imbalance in node *a*, we will *first* perform a *left rotation* of node e towards *b*.



Original tree

Fixing configuration LR

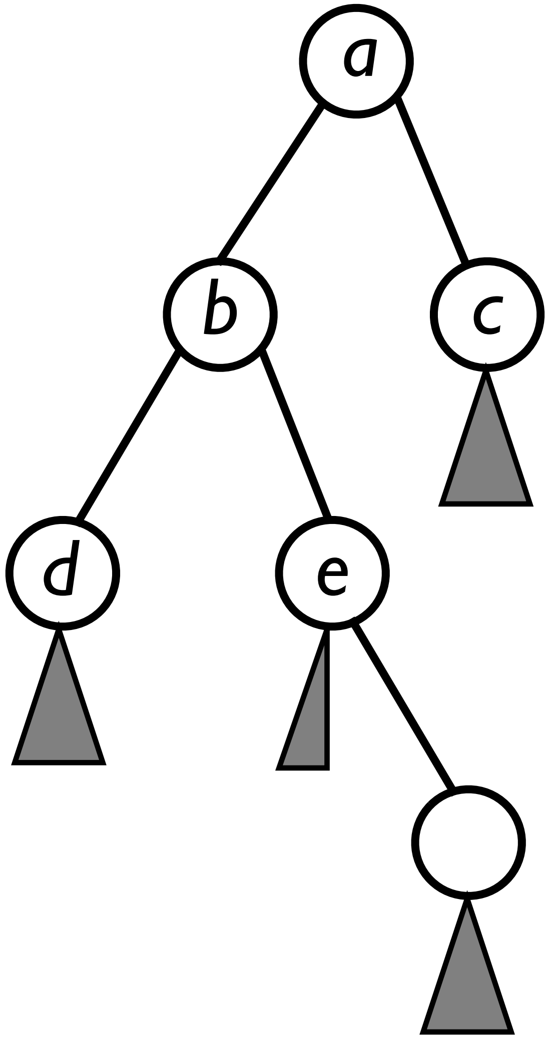
- To fix the imbalance in node *a*, we will *first* perform a *left rotation* of node e towards *b*.



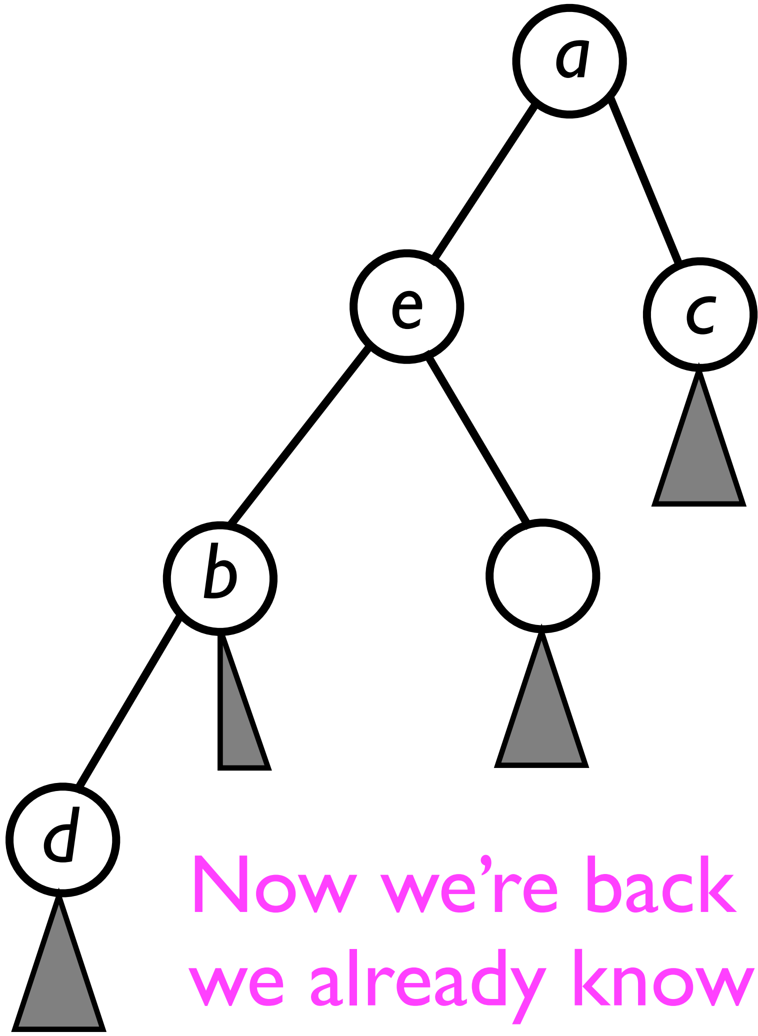
Original tree

Fixing configuration LR

- To fix the imbalance in node *a*, we will *first* perform a *left rotation* of node e towards *b*.



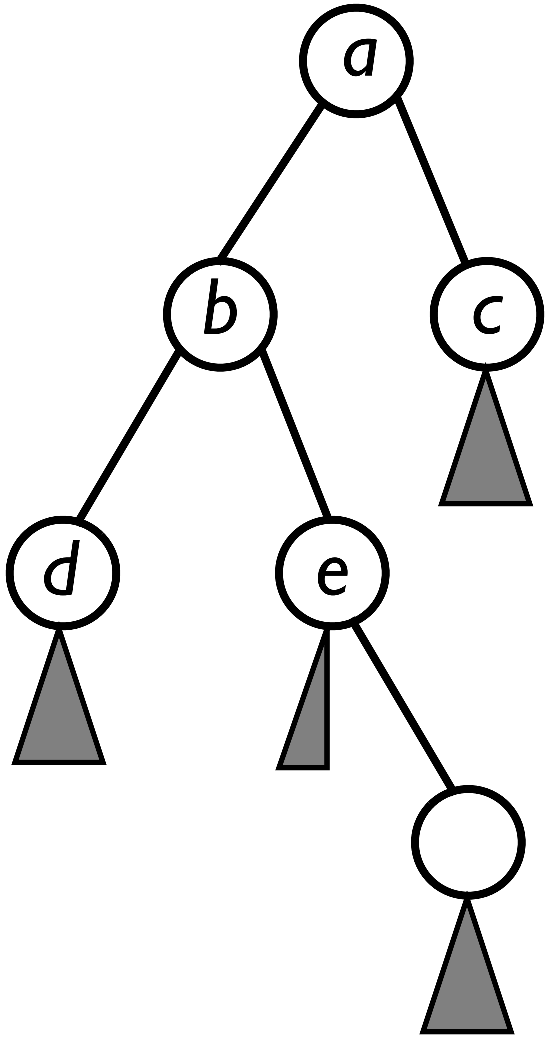
Original tree



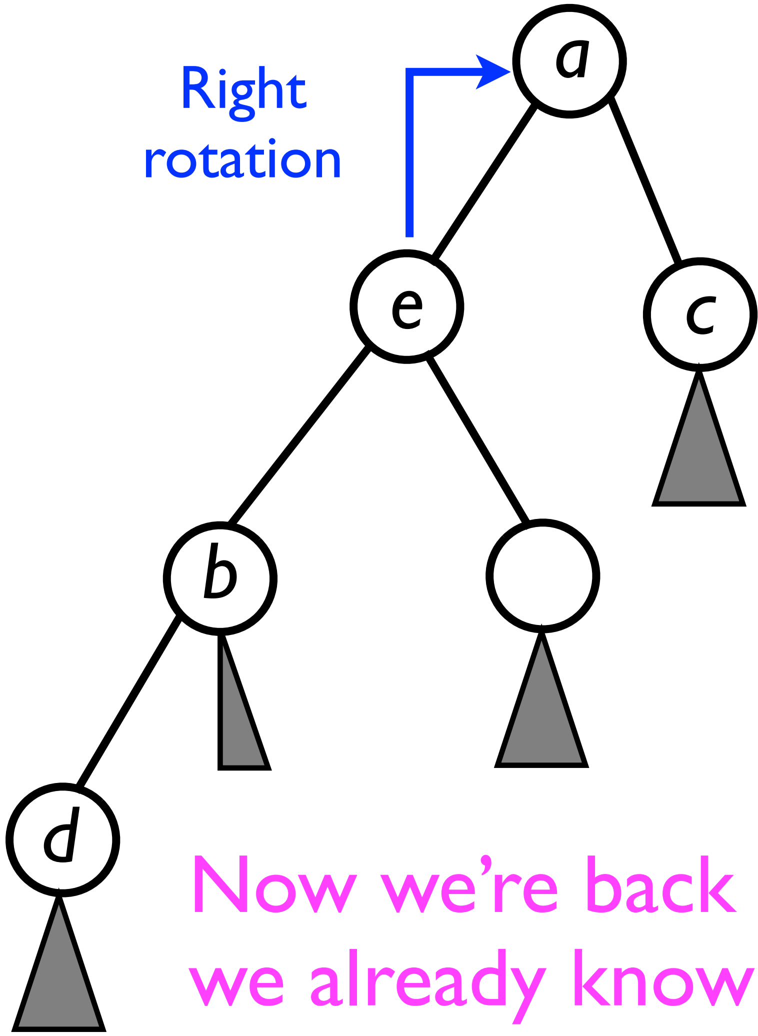
Now we're back to LL -- and we already know how to correct this (by applying a *right rotation of* e *towards* a).

Fixing configuration LR

- Now we perform a *right rotation* of e towards *a*.



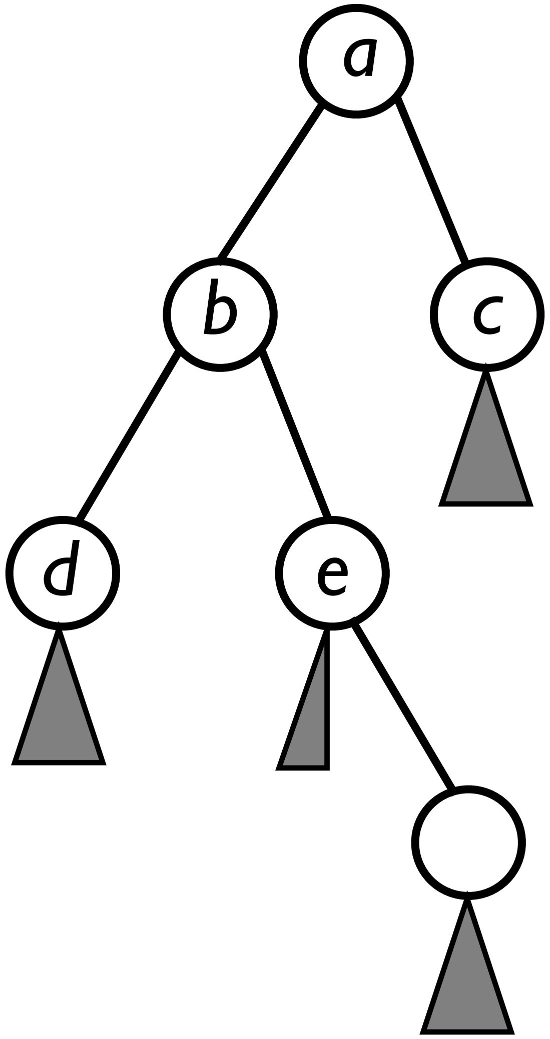
Original tree



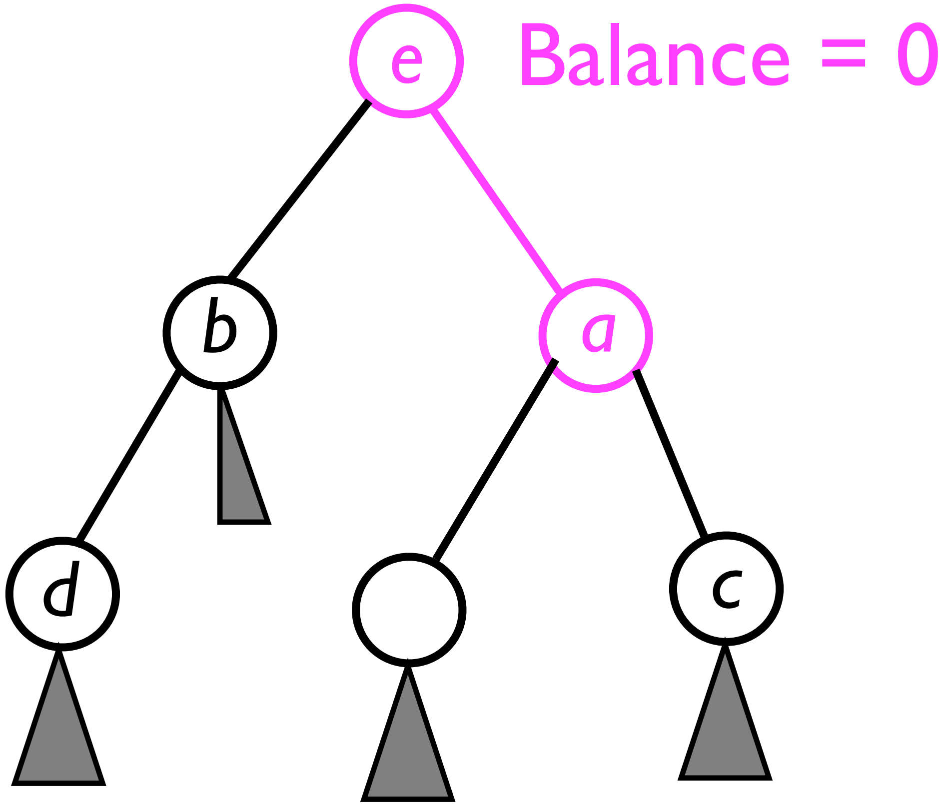
Now we're back to LL -- and we already know how to correct this (by applying a *right rotation of* e *towards* a).

Fixing configuration LR

- Now we perform a *right rotation* of e towards *a*.



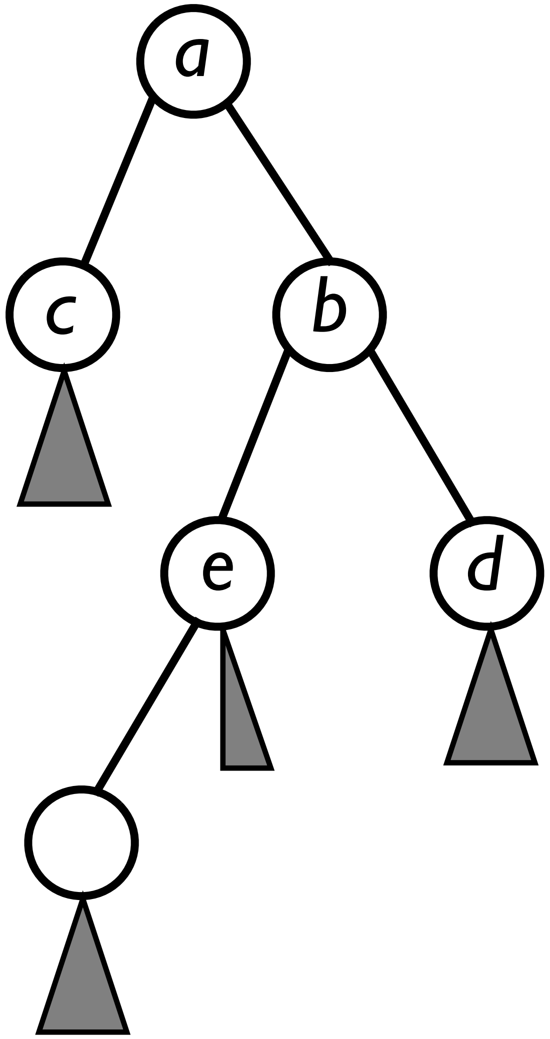
Original tree



Fixing configuration RL

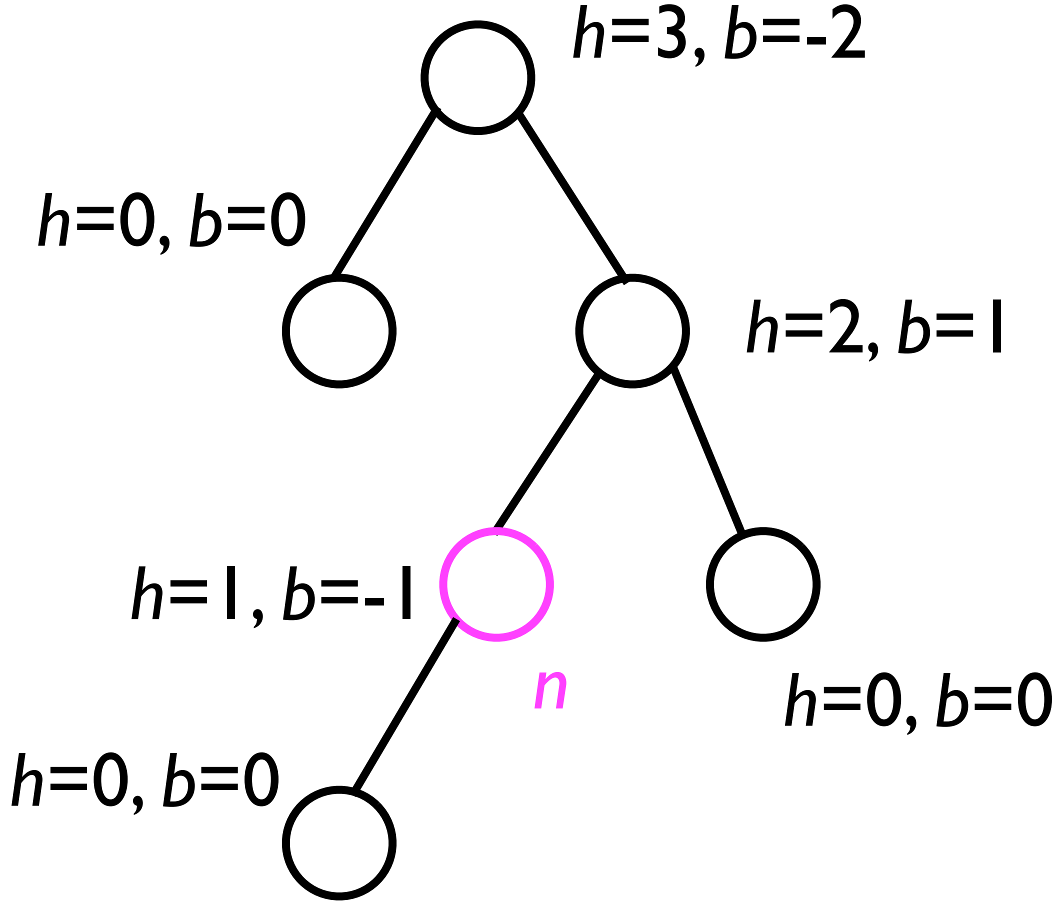
- Fixing configuration RL is exactly symmetric to fixing LR:
  - First apply a *right rotation of* e *towards* b.
    - This returns the configuration to RR.
  - Then apply a *left rotation of* e *towards* a.
  - Left as an "exercise for the reader".

Balance = -2



Removing a new node

- When we *remove* a node *n*, we must distinguish the three cases as outlined last lecture:
  - *n* is a leaf node.
  - *n* has only one child.
  - *n* has two children.
- After removing *n*, we must update the height and balance of all nodes between *n* and the root.



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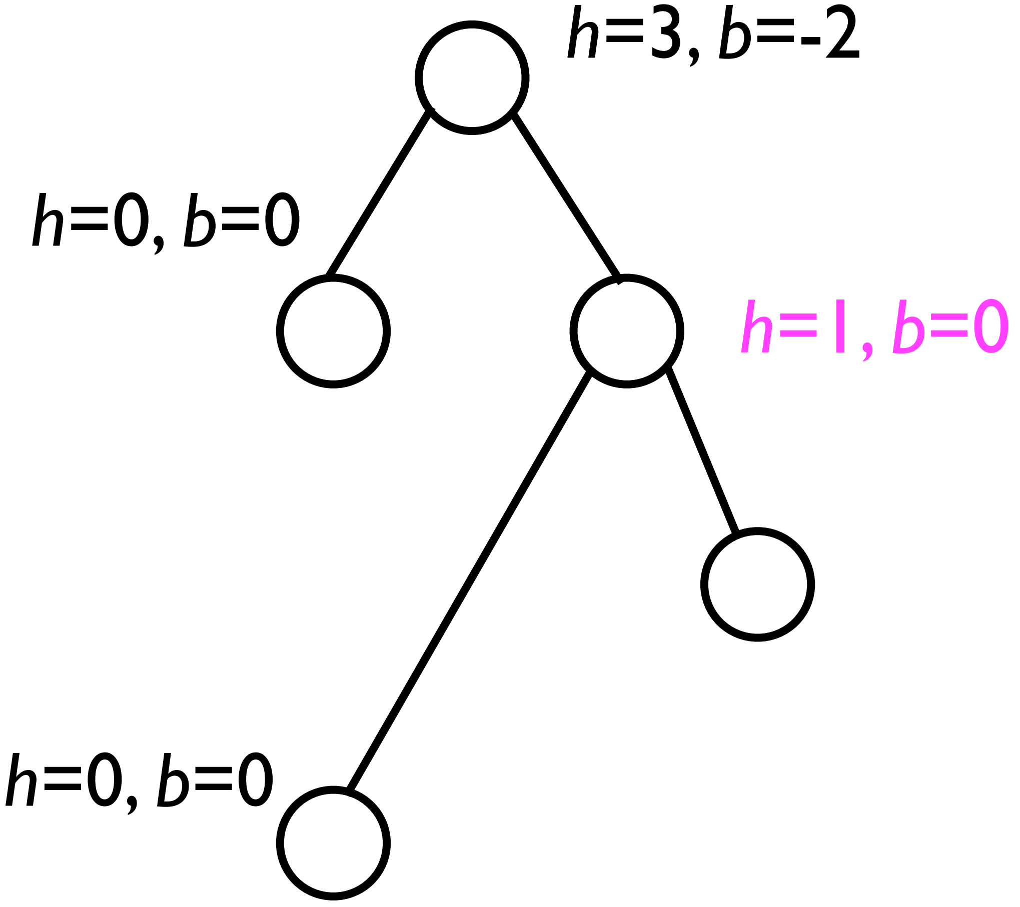
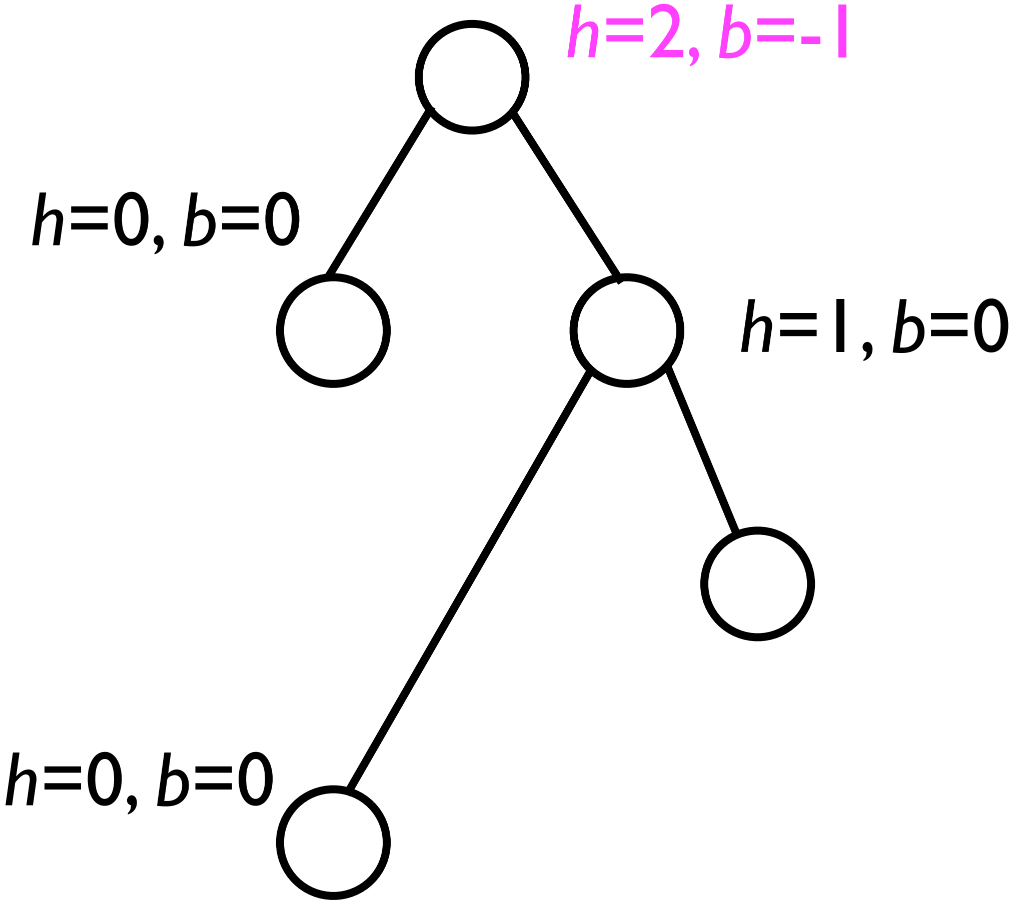


Removing a new node

- When we *remove* a node *n*, we must distinguish the three cases as outlined last lecture:
  - *n* is a leaf node.
  - *n* has only one child.
  - *n* has two children.
- After removing *n*, we must update the height and balance of all nodes between *n* and the root.
  - Might require an AVL rotation.



AVL trees

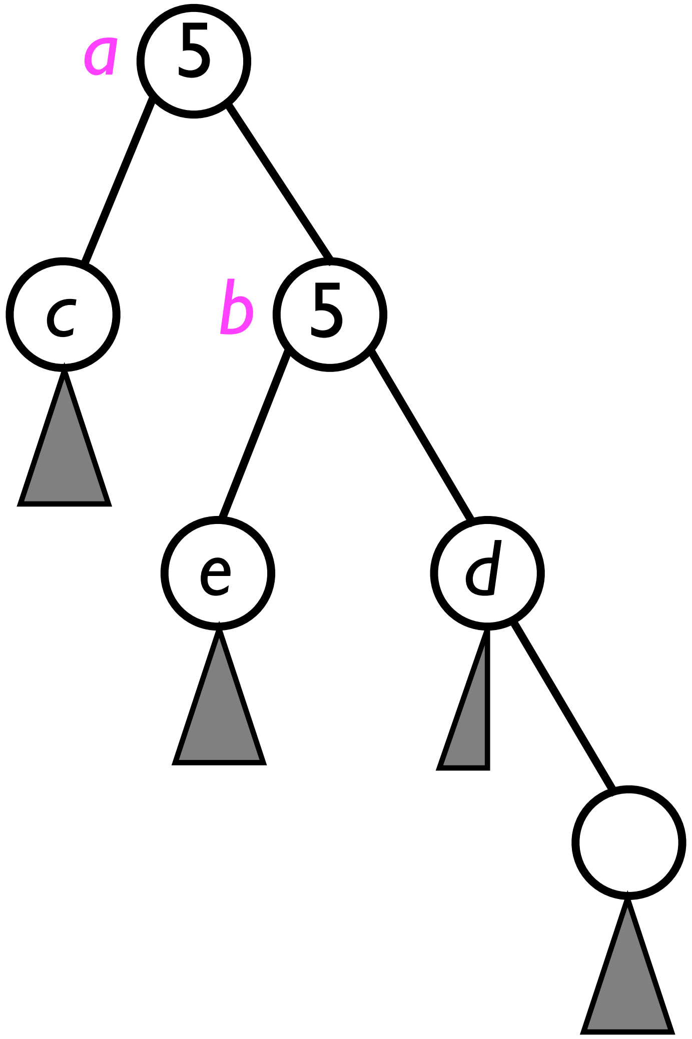
- Through storing the height and balance of each node and implementing AVL rotations as necessary, we can ensure that the BST is never “more imbalanced” than +1 or -1.
  - This yields a BST for which $h=O(\log n)$ in the *worst case*, not just the average case.
  - The AVL rotations themselves take O(1) time.
    - Each rotation takes a constant number of “node switches”.
  - Hence, with AVL trees, the fundamental tree operations **`add`**, **`find`**, and **`remove`** all operate in $O(\log n)$ time worst-case.

Duplicate keys

- In contrast to “regular” BSTs, duplicate keys are not permitted in AVL trees.
  - With duplicate keys, rotating sub-trees could cause the tree to violate the BST ordering property.

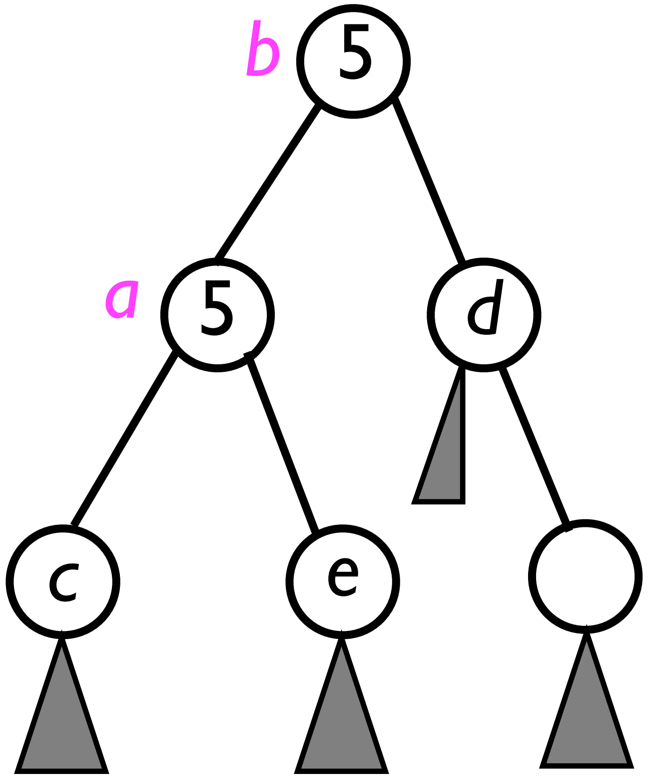
Duplicate keys

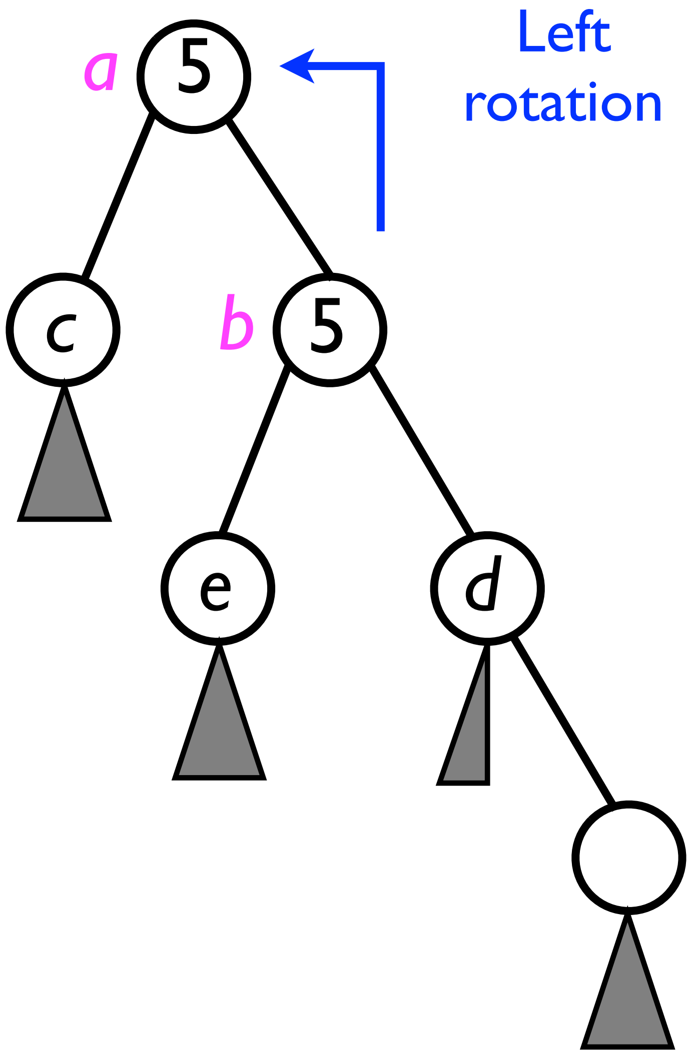
- However, a problem arises when we start rotating nodes in a sub-tree:
  - Suppose *a* and *b* have the same key (e.g., 5).



Duplicate keys

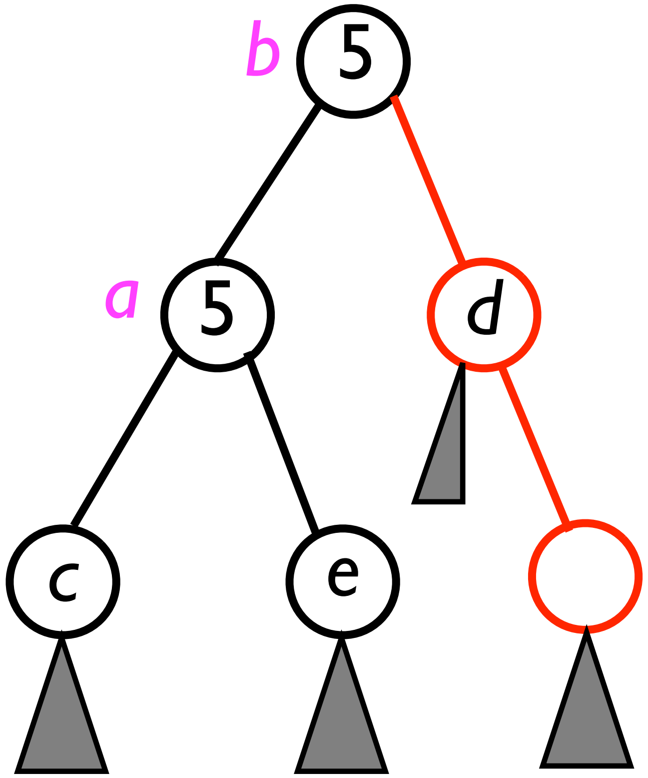
- However, a problem arises when we start rotating nodes in a sub-tree:
  - Suppose *a* and *b* have the same key (e.g., 5).
  - Suppose we then *left-rotate b* towards *a*.





Duplicate keys

- Now, suppose we want to find node *a* starting at the root (node *b*).
  - We will descend the *wrong sub-tree* of *b*.
  - We will never find *a*.



Duplicate keys

- One solution is to:
  - Disallow multiple *nodes* with the same key.
  - Whenever we add an element with the *same* key, we *append* that new element to that node's *list of objects*.