CSE 12:
Basic data structures and object-oriented design

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Lecture Eleven
24 July 2012

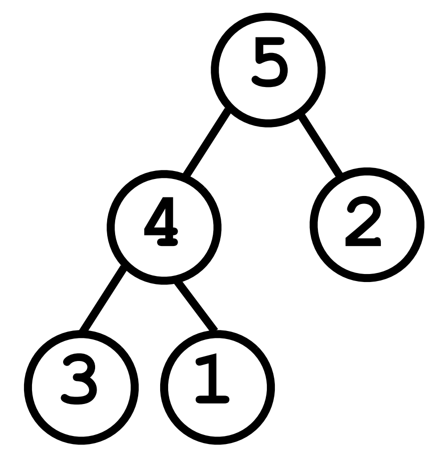
Heaps, continued.

Review from last lecture

- A *heap* is a *complete binary tree* whose last level of nodes is filled left-to-right *and* which satisfies the *heap condition*.
- Heap condition:
  - The root of every sub-tree is *no smaller than any node in the sub-tree*. (For *max*-heap).
- The heap condition ensures that the *largest* element is always stored at the root:
  - $O(1)$ time-cost for **`findLargest`**
  - $O(\log n)$ time-cost for **`removeLargest`**

Adding to a heap

- To add a new object o to the heap:
  - Create a new node *n* containing *o*, and add *n* to the last level of the tree (at the left-most position).
    - This may violate the heap condition.
  - Repeatedly "bubble up" n towards the root whenever *n* > parent(*n*).



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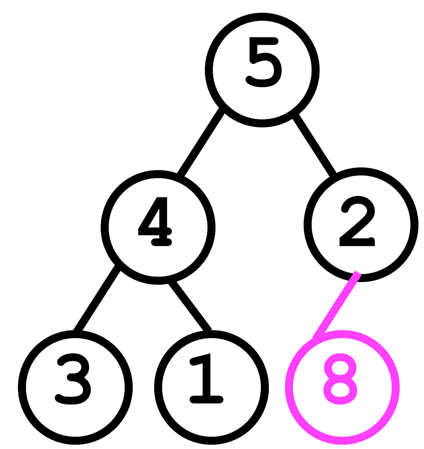

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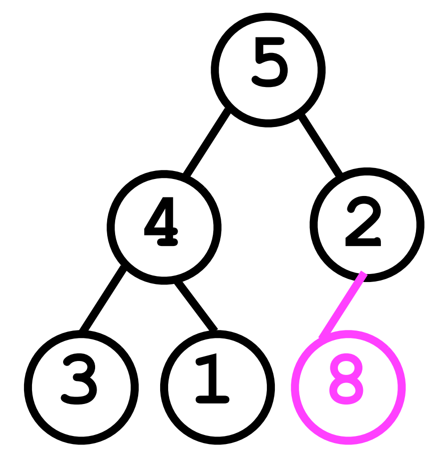

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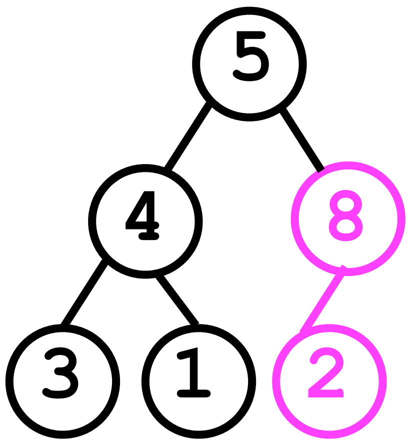

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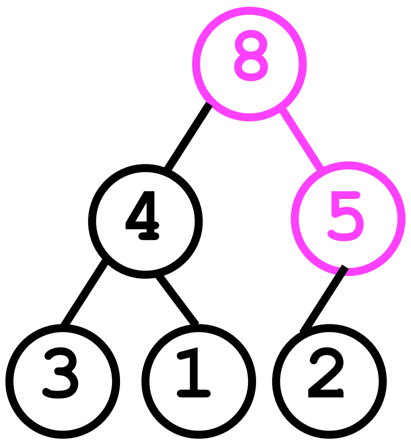

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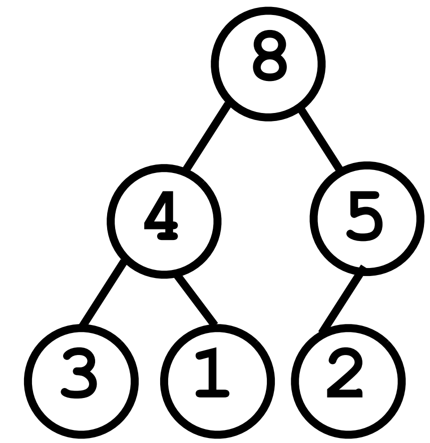

The tree is now a valid heap again.

Removing the *largest* element from a heap

- The largest element is always stored at the top of the heap.
  - Hence, just remove the *root*.
- We must then *replace* it with something.
  - Remove the last node *n* in the heap (right-most child of last level) and make it the new root of the tree.
    - This may violate the heap condition.
    - We will then have to recursively swap *n* with one of its children (i.e., back down the tree) until the heap condition is restored. This is called "trickling down".

Removing the *largest* element from a heap

```
void removeLargest () {
  _nodeArray[0] = _nodeArray[_numNodes - 1];
  _numNodes--;
  trickleDown(0);
}

void trickleDown (int index) {
  If node at index is less than one of its children:
    Swap node with the largest child node.
    trickleDown(largestChild(index));
}
```

Recursive implementation

or

```
void trickleDown (int index) {
  While node at index is less than one of its children:
    Swap node with the largest child node.
    index = largestChild(index);
}
```

Iterative implementation

Removing the *largest* element from a heap

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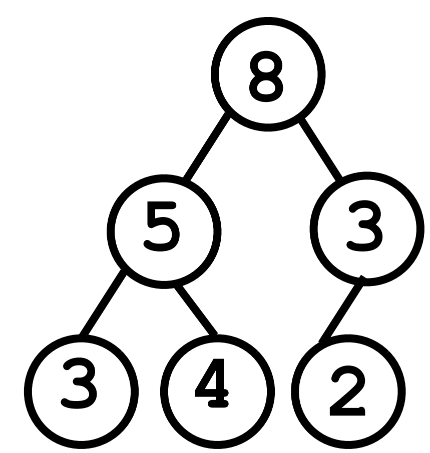

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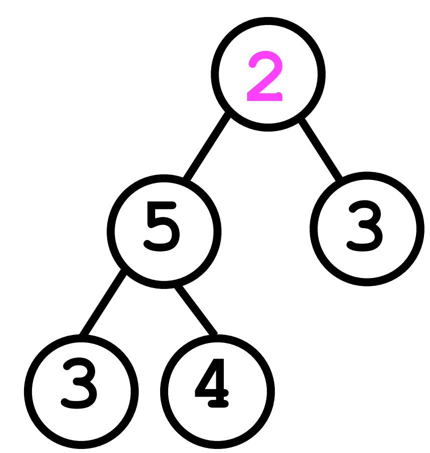

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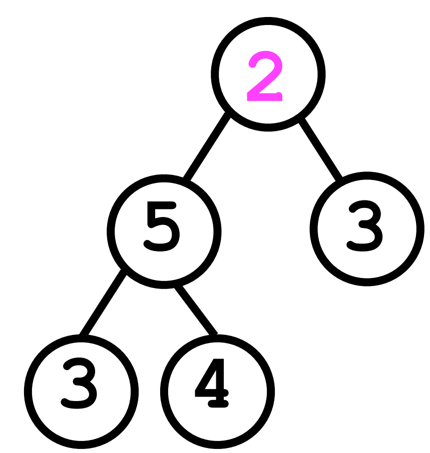

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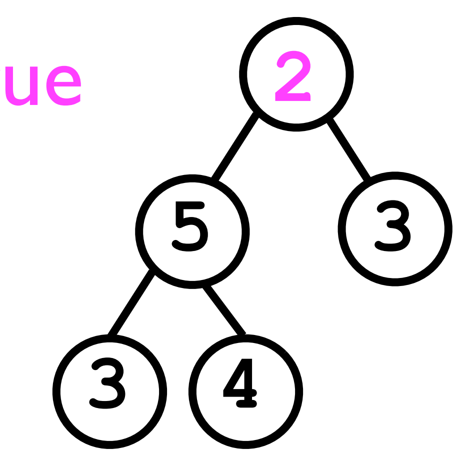

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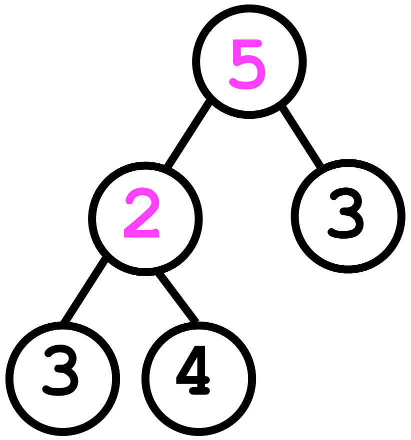

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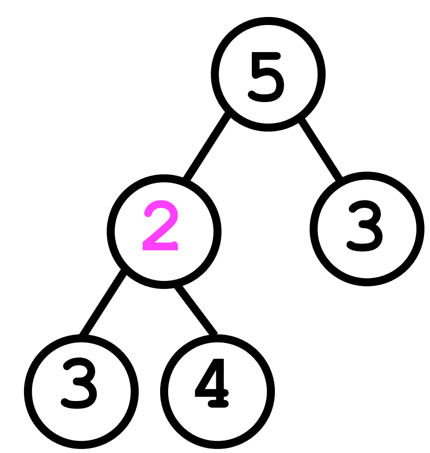

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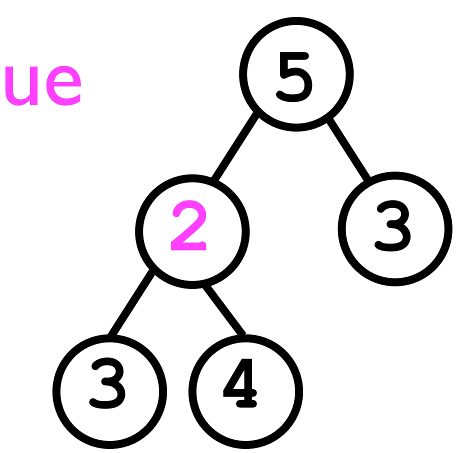

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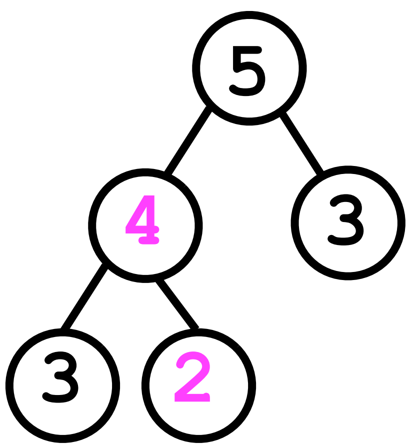
```

It's crucial we swap with the *larger* child to maintain the heap condition.



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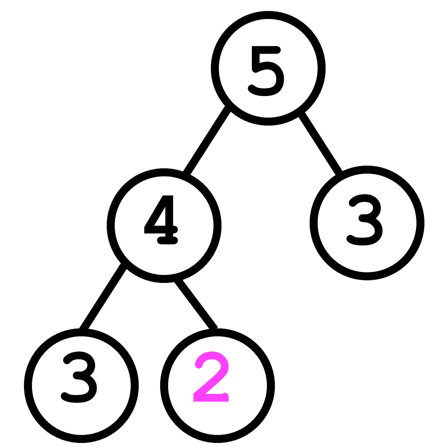

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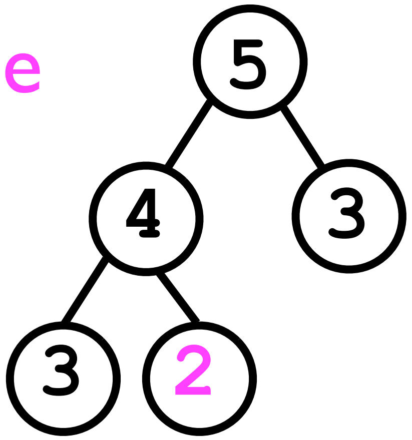

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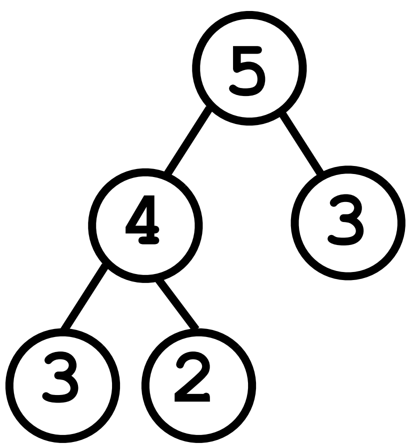
```

Done.



Finding an arbitrary node

- Heaps offer fast access to the *largest* node in the heap.
- However, despite their *binary tree* representation, they offer no advantage over simple *lists* in terms of finding an *arbitrary* element.
    - If the element o that the user wishes to find is not the largest, then o could be *anywhere* in the heap.
    - This contrasts with binary *search* trees (more later).
- Hence, to find an object o within a heap, we must search through the *entire heap*.

Finding an arbitrary node

```
public T find (T o) {
  final int index = findNode(0, o);
  if (index < 0) {
    throw new NoSuchElementException();
  }
  return _nodeArray[index];
}

private int findNode (int rootIdx, T o) {
  if (_nodeArray[rootIdx].equals(o)) {
    return rootIdx;
  }

  int idx;
  if (leftChild(rootIdx) < _numNodes &&
      (idx = find(leftChild(rootIdx), o)) >= 0) {
    return idx;
  } else if (rightChild(rootIdx) < _numNodes &&
      (idx = find(rightChild(rootIdx), o)) >= 0) {
    return idx;
  } else {
    return -1;
  }
}
```

We *could* implement `findNode` by recursively searching through the entire tree.

Finding an arbitrary node

But this is much easier (and slightly faster too).

```
int findNode (T o) {
  for (int i = 0; i < _numNodes; i++) {
    if (_nodeArray[i].equals(o)) {
      return i;
    }
  }
}
```

- This is one of the conveniences of representing the tree as an array.
  - Only possible for *complete* trees in which there are no “holes” in the array (i.e., missing child nodes).

Removing an arbitrary node

- Removing an arbitrary node requires that we first *find* the node *n* to be removed.
  - We can use the `findNode(o)` method we just constructed.
- Once found, we can *swap* the *last* node in the heap (right-most child of last level) with *n*.
- Then we just `trickleDown` that node and we're done, right?

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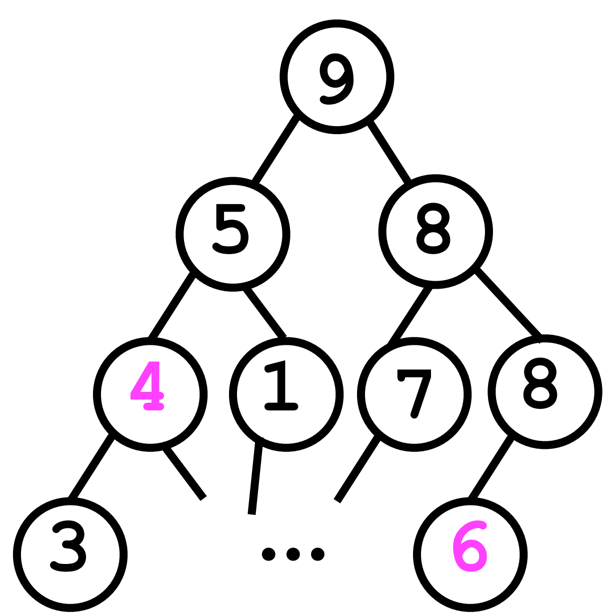
- Once found, we can *swap* the *last* node in the heap (right-most child of last level) with *n*.
- Then we just **`trickleDown`** that node and we're done, right? Wrong.

Removing an arbitrary node

- The above procedure worked for `removeLargest()` because we always started from the *top* (root) of the heap.
  - By trickling down from the top, we guarantee that *every* sub-tree (starting from the very top) is a valid heap.
- When removing an *arbitrary* node, the `trickleDown` process will "fix" the sub-tree rooted at *n*, but *not necessarily* the *whole* tree.
- What's an example heap in which this problem would arise?

Removing an arbitrary node

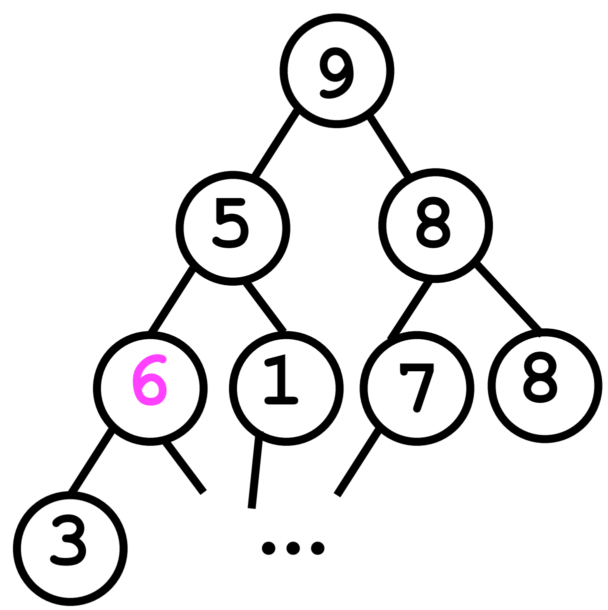
- Suppose we wish to remove the node containing 4.
- If we just replace it with the "last" node (6)...



Valid heap.

Removing an arbitrary node

- ...then the **`trickleDown()`** method will do nothing (6 is already bigger than its children).
- Moreover, 6 is now bigger than its parent -- a *violation of the heap condition*.



Invalid heap.

Removing an arbitrary node

- In a correct implementation of **`remove(o)`** for arbitrary o, we need to *sometimes* **`bubbleUp`** *and sometimes* **`trickleDown`**:

```
void remove (T o) {
  Find the node n containing o.
  Replace n with the “last” node l in the heap.
  If l < n:
    trickleDown on n.
  Else:
    bubbleUp on n.
}
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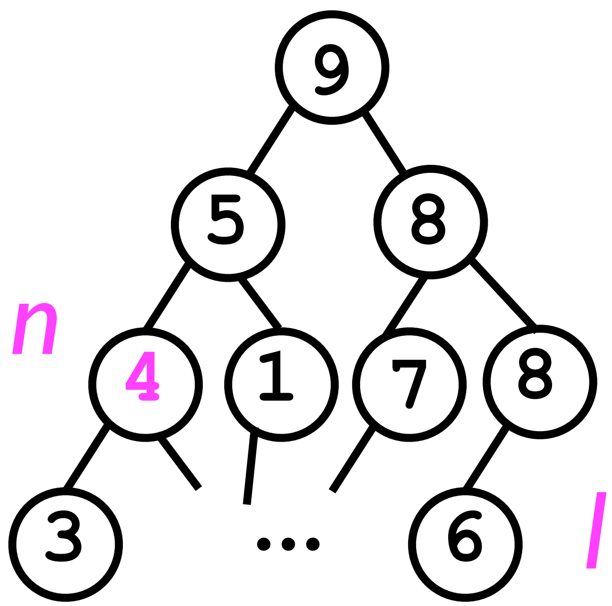

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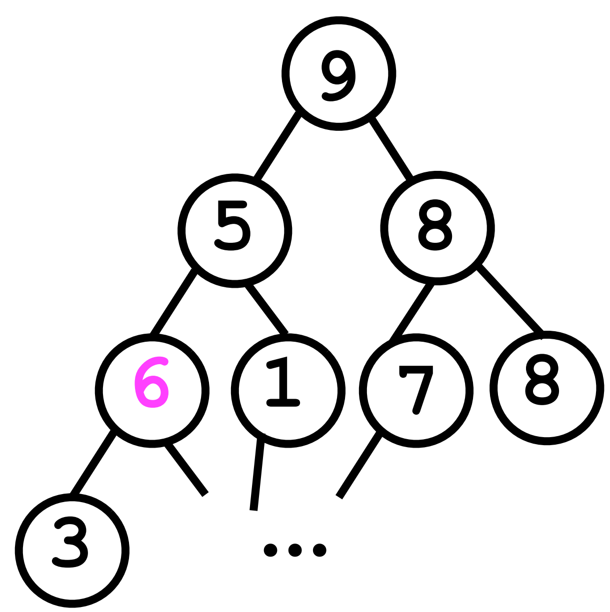

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void remove (T o) {
  Find the node n containing o.
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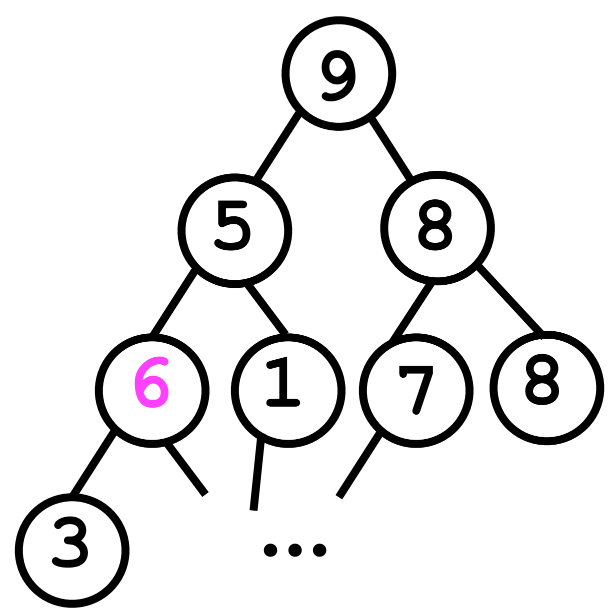

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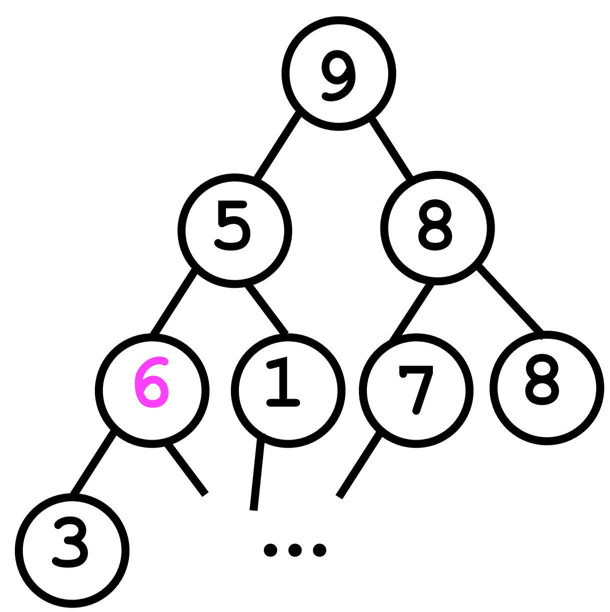

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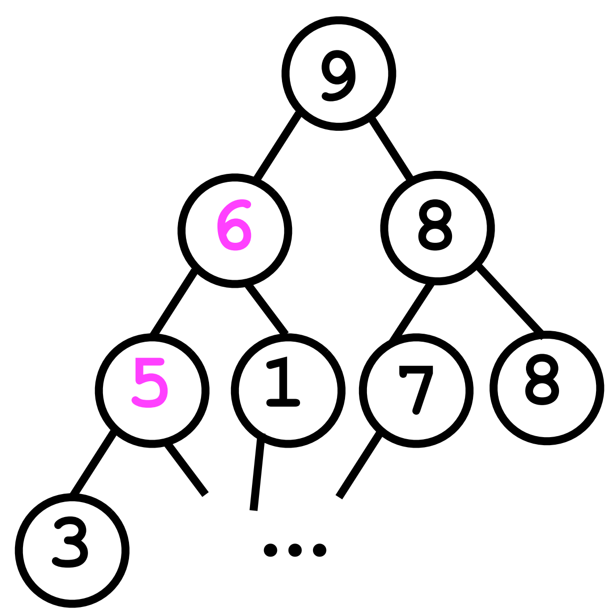

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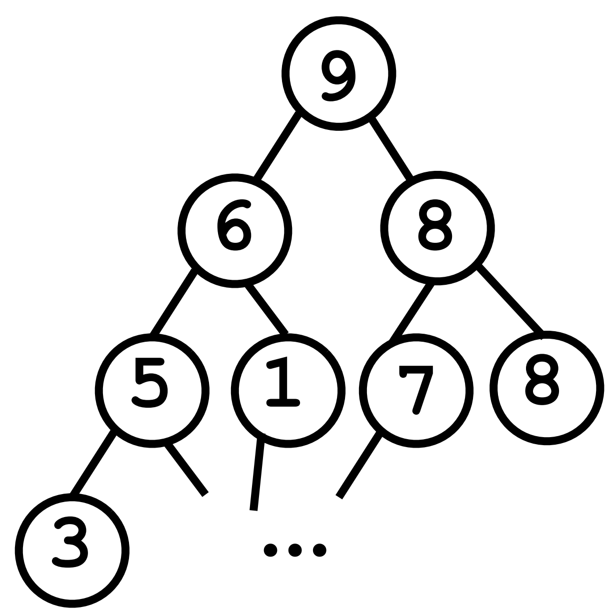


Valid heap again.

Heap operations: time costs

- The implementations for the **add**/**find**/**removeLargest**/**remove** methods depend on the methods **bubbleUp** and **trickleDown**.
- ```
  void bubbleUp (int idx) {
    While node at idx is "larger" than its parent:
      Swap data in the node and its parent;
      Set idx to be parentIdx(idx);
  }
  ```
  - At each loop iteration, **idx** moves one step closer from a leaf to the root of the heap.
    - Hence, loop can execute maximum of *h* times (*h* is tree height). For heap of *n* nodes, *h* is $\log_2(n)$.
  - Inside loop, the time cost is about 2 operations.
  - Hence, time cost is $O(\log n)$.
  ```

Heap operations: time costs

- ```
  void trickleDown (int index) {
    While node at index is less than one of its children:
      Swap node with the larger child node.
      index = largerChild(index);
  }
  ```
  - At each loop iteration, **`idx`** moves one step closer from the root of the heap to a leaf.
    - Hence, number of iterations is bounded by $h = \log_2(n)$.
  - Inside loop, the time cost is about 2 operations.
  - Hence, time cost is $O(\log n)$.
  ```

Heap operations: time costs

- Given the time costs of **`bubbleUp`** and **`trickleDown`**, we can compute the worst-case time costs of the fundamental heap operations:
  - **`add(o)`**: $O(1)+O(\log n) = O(\log n)$
    - Append a new node to the heap. $O(1)$
    - Bubble it up. $O(\log n)$
  - **`removeLargest()`**: $O(1)+O(\log n) = O(\log n)$
    - Swap last node with root. $O(1)$
    - Trickle root down. $O(\log n)$

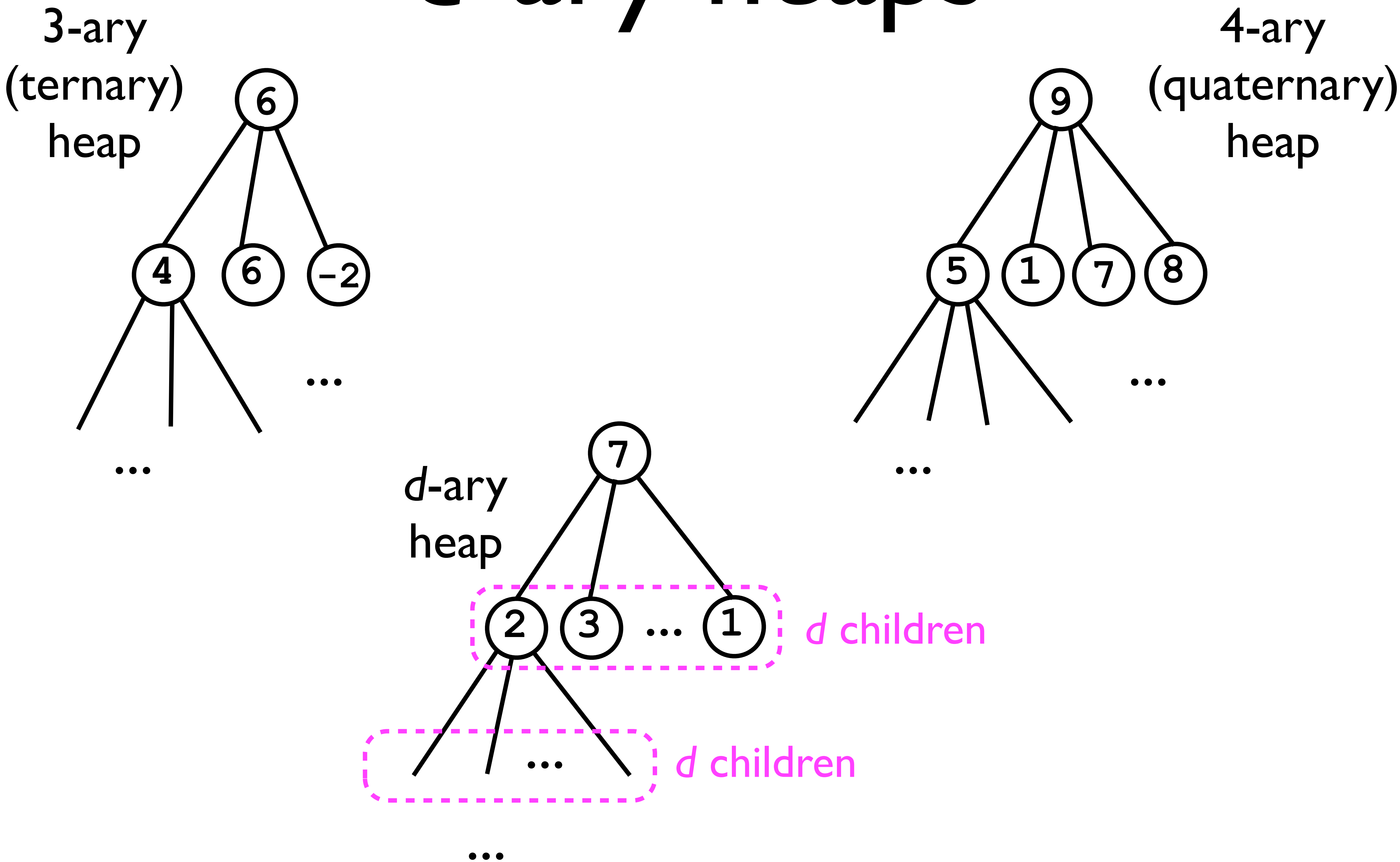
Heap operations: time costs

- `find(o)`: $O(n)$
  - Search through all nodes. $O(n)$
- `remove()`: $O(n)+O(1)+O(\log n) = O(n)$
  - Find the node. $O(n)$
  - Swap node-to-remove with root. $O(1)$
  - *Either* trickle node down *or* bubble it up. $O(\log n)$

General heaps

- We have just described the minimal implementation of a *binary heap*.
  - Binary heaps are the most common.
- In theory, however, *any* tree can be a heap as long as it satisfies the *heap condition* that the root of every sub-tree is no smaller than any node in the sub-tree.
- In particular, we can define a *d*-ary tree in which each node has *d* child nodes (instead of always 2).

d-ary heaps



3-ary
(ternary)
heap
6
4
6
−2
...
...
4-ary
(quaternary)
heap
9
5
1
7
8
...
...
d-ary
heap
7
2
3
...
1
d children
...
d children
...

d-ary heaps: Why?

- *d*-ary heaps can offer a time cost savings compared to binary heaps.
- Consider:
  - The height *h* of a binary heap is at most $\log_2(n)$.
  - The height *h* of a ternary heap is at most $\log_3(n)$.
  - The height *h* of a d-ary heap is at most $\log_d(n)$.
- As the *base* of the logarithm (*d*) gets *larger*, the *value* of the logarithm itself grows *smaller*.
- Hence, for larger *d*, operations that depend on the *height* of the tree will become *faster*.

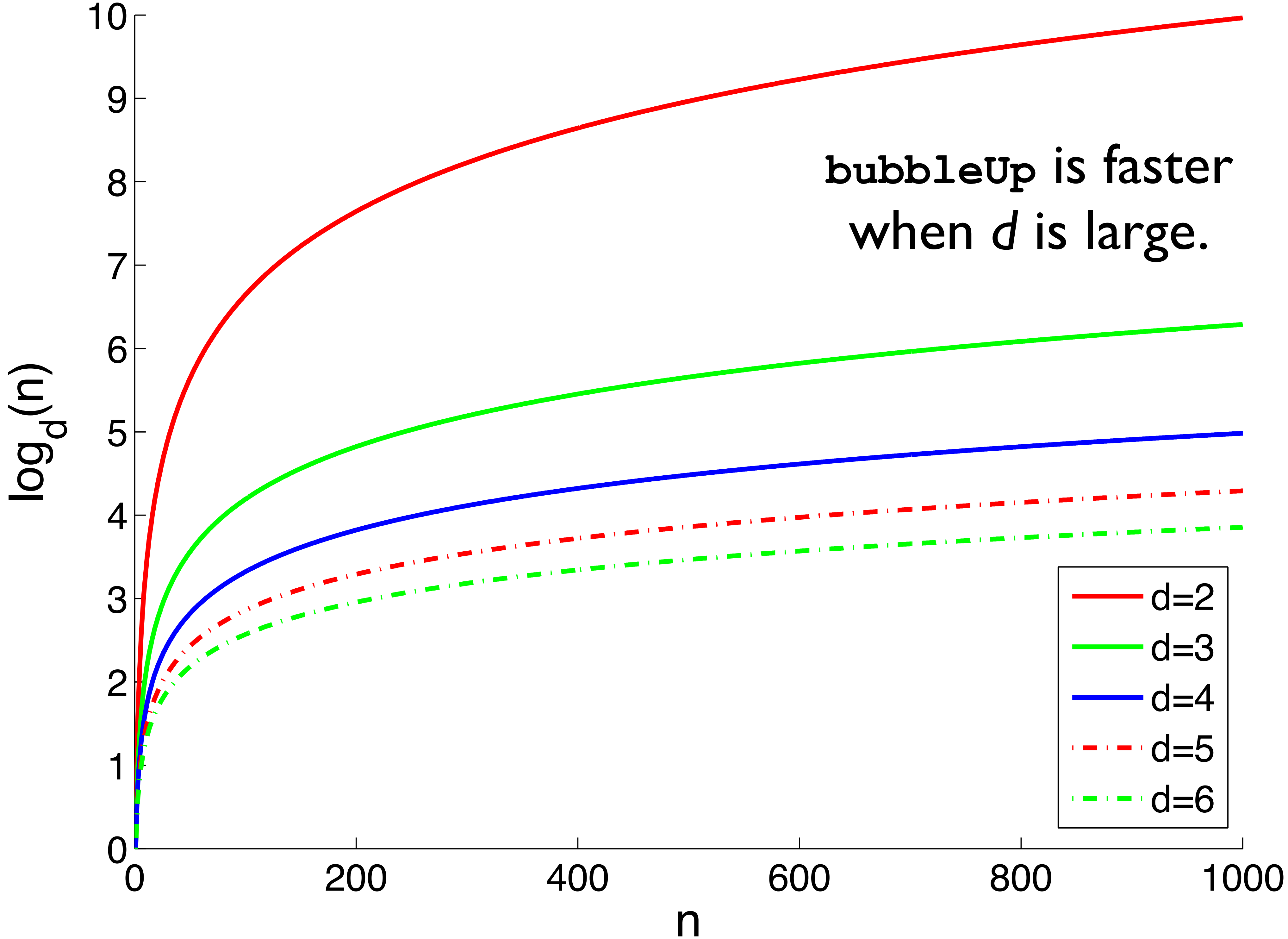
d-ary heaps: Why?

- On the other hand, as *d* increases, so does the *number of children per node.*
- The time cost of **`trickleDown`** (but not **`bubbleUp`**) is affected by the *number* of children:

```
void trickleDown (int index) {
  While node at index is less than one of its children:
    ...
}
```

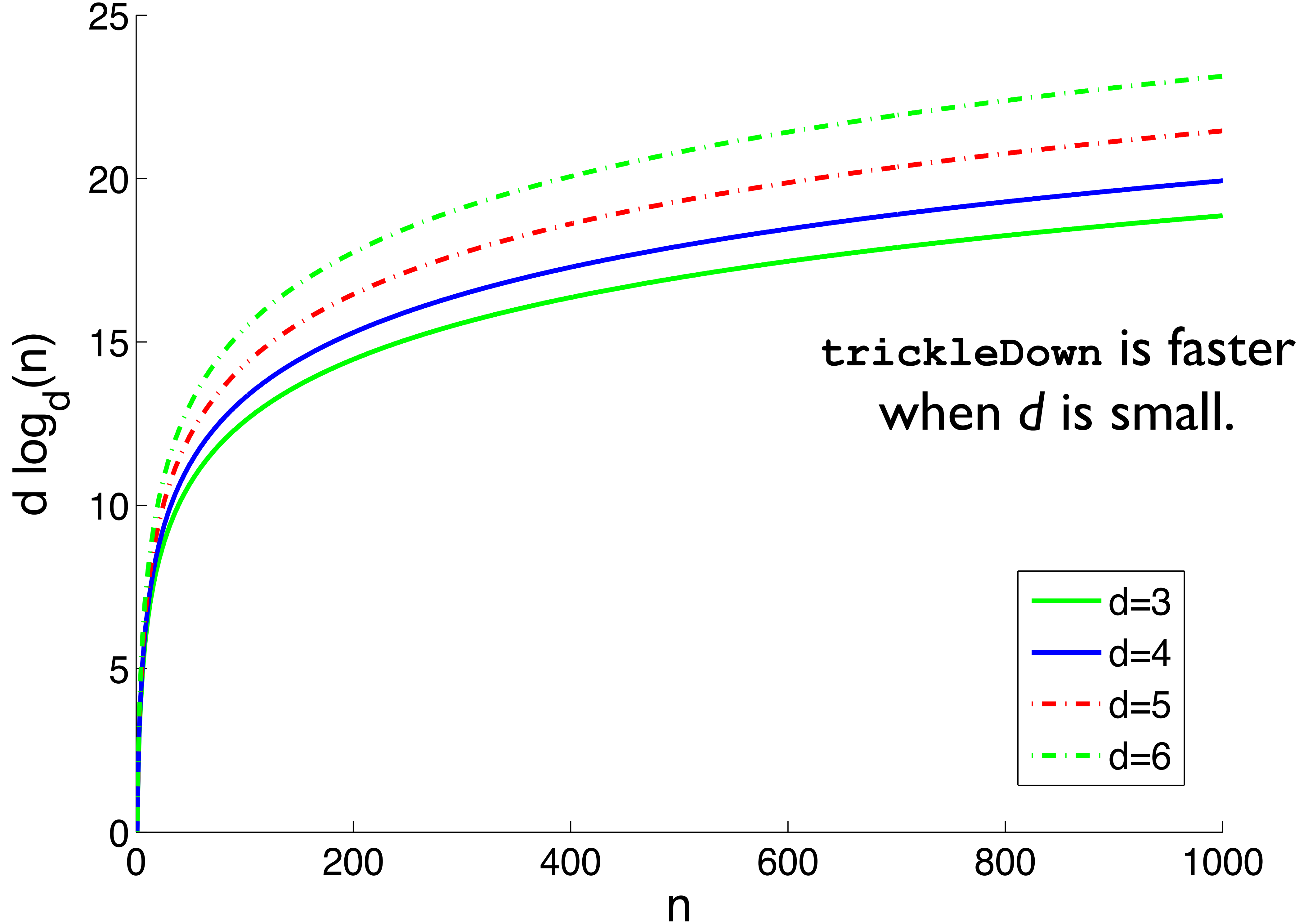
  - Each loop iteration implicitly requires a comparison to all *d* children.
  - The loop runs for at most *h* iterations ($h = \log_d n$), and each iteration takes at least *d* operations.
  - Hence, time cost for **`trickleDown`** is $O(hd) = O(d \log_d n)$.

`bubbleUp`: $O(\log_d n)$



`bubbleUp` is faster when *d* is large.

`trickleDown`: $O(d \log_d n)$



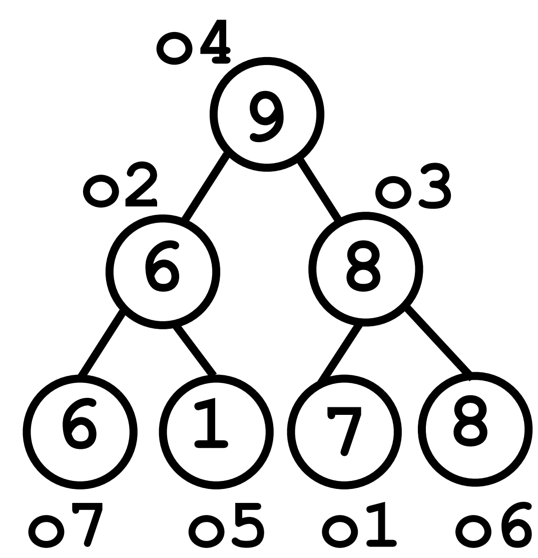
`trickleDown` versus `bubbleUp`

- In scenarios where `bubbleUp` is called more frequently than `trickleDown`, better time costs can be achieved using a larger value of *d*.
    - Such scenarios can happen with *priority queues* when the user *changes the priority* of the data while they are still in the heap.

Increasing/decreasing priority

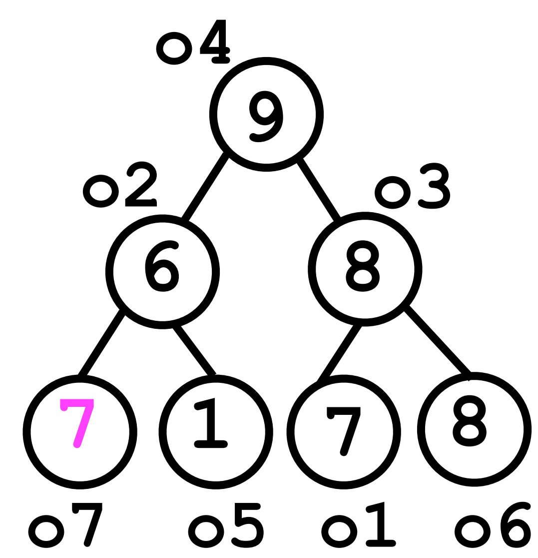
- Example:

```
heap.add(o1);  // Priority 7
heap.add(o2);  // Priority 6
...
heap.add(o7);  // Priority 6
```



Increasing/decreasing priority

- Example:
```
heap.add(o1);  // Priority 7
heap.add(o2);  // Priority 6
...
heap.add(o7);  // Priority 6
```
- Later on:
```
heap.increasePriority(o7);
```



Now we need to **bubbleUp** o7.

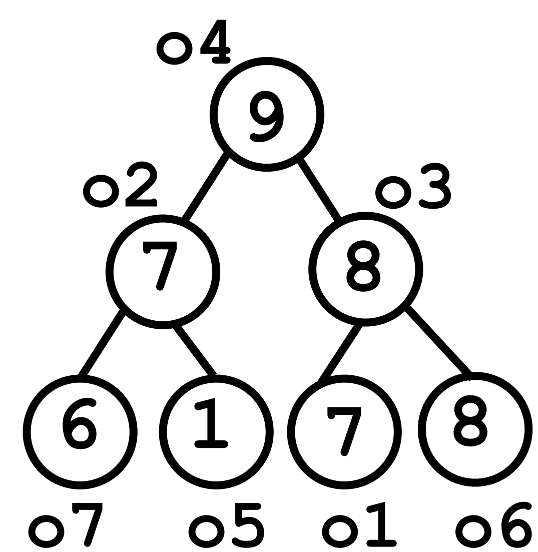
Increasing/decreasing priority

- Example:

```
heap.add(o1);  // Priority 7
heap.add(o2);  // Priority 6
...
heap.add(o7);  // Priority 6
```

- Later on:

```
heap.increasePriority(o7);
```



Done.

`trickleDown` versus `bubbleUp`

- *Increasing* the priority of an item requires `bubbleUp` to be called to maintain the heap condition.
- *Decreasing* the priority of an item requires `trickleDown` to be called to maintain the heap condition.
- In some applications, the user may want to *increase* the priority of items more frequently than they will *decrease* their priority.
  - In this case, `bubbleUp` will be called more frequently than `trickleDown`.
  - By using a *d*-ary heap and setting $d>2$, the time cost of the priority queue may be reduced compared to a binary heap.