Sequential Least Squares

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Let
\[ \rho(\theta,t) = (1 - \gamma)\left( \sum_{s=0}^{\infty} \gamma^s e^2(\theta,s) \right) \]  
(1)

where
\[ e^2(\theta,s) = \frac{1}{2} (y_s - x'_s \theta)^2 \]  
(2)

Note
\[ \frac{\partial e^2(\theta,s)}{\partial \theta} = x_s y_s - x_s x'_s \theta \]  
(3)

and
\[ \frac{\partial \rho(\theta,t)}{\partial \theta} = (1 - \gamma)\left( \sum_{s=0}^{\infty} x_s y_s \right) - (1 - \gamma)\left( \sum_{s=0}^{\infty} x_s x'_s \right) \theta \]  
(4)

\[ = f_{xy}(t) - f_{xx}(t) \theta \]  
(5)

where
\[ f_{xx}(t) = (1 - \gamma)\left( \sum_{s=0}^{\infty} \gamma^s x_s x'_s \right) \]  
(6)

\[ f_{xy}(t) = (1 - \gamma)\left( \sum_{s=0}^{\infty} \gamma^s x_s y_s \right) \]  
(7)

The minimum or \( \rho(\theta,t) \) with respect to \( \theta \) is achieved for
\[ \theta_t = f_{xx}^{-1}(t)f_{xy}(t) \]  
(9)

Note the \( f_{xx} \) matrix and the \( f_{xy} \) vector can be computed in a recursive manner
\[ f_{xx}(t) = (1 - \gamma)x_t x'_t + \gamma f_{xx}(t-1) \]  
(10)

\[ f_{xy}(t) = (1 - \gamma)x_t y_t + \gamma f_{xy}(t-1) \]  
(11)

Moreover, the inverse matrix \( f_{xx}^{-1}(t) \) can also be computed in a recursive manner. Using the matrix inversion theorem
\[ f_{xx}^{-1}(t) = \gamma \left( f_{xx}^{-1}(t-1) - \frac{\alpha}{1 + \alpha x'_t z_t} z_t z'_t \right) \]  
(13)

where
\[ \alpha = \frac{1 - \gamma}{\gamma} \]  
(14)

\[ z_t = f_{xx}^{-1}(t-1)x_t \]  
(15)

1 Appendix: Example Code

```matlab
% Example Matlab Code for Sequential Least Squares
% Javier R. Movellan. Feb 2010
clear
```
p = 10; % The dimensionality of the input
T = 500; % The number of time steps
gamma = 0.99;
alpha = (1-gamma)/gamma;

trueTheta = (1:p)';
x = randn(p,T);
y = x'*trueTheta;

theta = zeros(p,T); % our running estimate of theta;
fxxinv = eye(p); % Initial value of the fxxInv matrix
fxy = zeros(p,1); % Initial value of the fxy vector
theta(:,1) = fxxinv*fxy;

for t=2:T
    xt = x(:,t);
    zt = fxxinv*xt;
    fxy = (1-gamma)*xt*y(t)+ gamma*fxy;
    fxxinv = (fxxinv - (alpha/(1+ alpha*xt'*zt))*zt*zt')/gamma;
    theta(:,t) = fxxinv*fxy;
end
plot(theta')
xlabel('Trial Number');
ylabel( '\theta');