Primer on POMDPs and Infomax Control

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1 MDP Finite Horizon Problems

For more detailed specification of the notation standards see the Appendix. We use a Matlab-style convention to denote sequences. Under this convention $x_{t:T} = (x_t, \cdots, x_T)$. We use capital letters for random variables and small letters for specific values taken from those variables. A process is a collection of variables indexed by $t = 1, \cdots, T$, where T is called the *horizon*, or *terminal time*. The processes of interest are:

- *System Process*: $X = \{X_1, \cdots, X_T\}$. Where the system, or state variables X_t take values in $\{1, \cdots, n_x\}$
- *Action Process*: $U = \{U_1, \cdots, U_T\}$ where the actions U_t take values in $1, \cdots, n_u\}$.
- *Control Process*: $C = \{C_1, \cdots, C_t\}$. Where each $C_t : H_t \to U_t$ maps states into actions.
- *Reward Process*: $R = \{R_1, \cdots, R_T\}$. Where $R_t : (X_t, U_t) \to \Re$ maps state, action combinations into real valued numbers.
- *Return Process*: $\bar{R} = \{\bar{R}_1, \cdots, \bar{R}_T\}$. Where $\bar{R}_t \stackrel{\text{def}}{=} \sum_{\tau=t}^{T} \gamma^{\tau - t} R_\tau$, and $\gamma \in [0, 1]$ is called the *discount factor*.
- *Value function for a given controller c:* It provides the expected return given that we visit state x_t at time t and use controller c to map states into actions.

$$V_t^c(x_t) = E[\bar{R}_t \mid x_t, c] \tag{1}$$

- *Optimal value function:* It provides the expected return given that we visit state x_t at time t, optimized over possible controllers

$$V_t(x_t) = \max_c E[\bar{R}_t \mid x_t, c] \tag{2}$$

- *State/Action value function under controller c:*

$$V_t^c(x_t, u_t) = E[\bar{R}_t \mid x_t, u_t, c] \tag{3}$$

  Note we are overloading the symbol for function V^c and identifying the function by the number of arguments.
- *Optimal state/action value Function:*

$$V_t(x_t, u_t) = \max_c E[\bar{R}_t \mid x_t, u_t, c] \tag{4}$$

- Generative Model (See Figure 1).

  (X_t, C_t) generate U_t.
  (X_t, U_t) generate (R_t, X_{t+1}).
- *System matrices:* $a = \{a^1, \cdots, a^{n_u}\}$, where a^u is an $n_x \times n_x$ matrix with

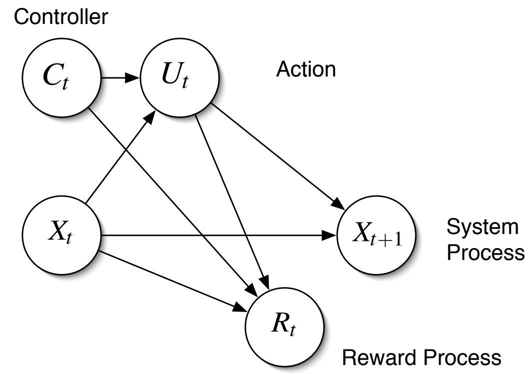
$$a^u_{i,j} = p(X_{t+1} = j \mid X_t = i, U_t = u) \tag{5}$$

- *Reward vectors:* r_t^u is an n_x dimensional vector such that

$$r^u_{t,i} = E[R_t \mid X_t = i, U_t = u] \tag{6}$$

Remark 1.1. Alternative Conventions: Some documents (e.g., Thrun Probabilistic Robotics) use the convention that C_t maps X_t into U_{t+1}. This does not affect the main results presented here, except for the shift in notation.

Figure 1: *Graphical Representation of a time slice of the process under study. Arrows represent dependency relationships between variables. An arrow from variable X to variable Y indicates that X is a* "parent variable" *of Y.*



Our goal is to find a controller c that maximizes the return. While this appears as a difficult optimization problem it has rich structure that allows solving it in a relatively efficient manner. Key to this solution is the *Optimality Theorem*, described below. To prove that theorem we will first need a lemma that tells us a condition under which the maximum of a sum of function equal the sum of the maxima.

Lemma 1.1 (Max Sum Lemma). *Let $w_i \geq 0$, for $i = 1, \cdots, n$ and let $\hat{x}$ maximize $f_i(x)$ for $i = 1, \cdots, n$, i.e.,*

$$f_i(\hat{x}) = \max_i f_i(x) \tag{7}$$

for $i = 1, \cdots, n$. Then

$$\max_x \sum_i w_i f_i(x) = \sum_i w_i f_i(\hat{x}) \tag{8}$$

Proof. Since $w_i \geq 0$ it follows that

$$\max_x \sum_i w_i f_i(x) \leq \sum_i \max_x w_i f_i(x) = \sum_i w_i f_i(\hat{x}) \tag{9}$$

Moreover, by definition of the maximum operation, it follow that for any value u

$$\max_x \sum_i w_i f_i(x) \geq \sum_i w_i f_i(u) \tag{10}$$

Thus, choosing $u = \hat{x}$,

$$\max_x \sum_i w_i f_i(x) \geq \sum_i w_i f_i(\hat{x}) \tag{11}$$

□

Theorem 1.1 (Optimality Theorem). *Let $\hat{c}_{t+1:T}$ be a controller that maximizes $E[\bar{R}_{t+1} \mid x_{t+1}, c_{t+1:T}]$ for all x_{t+1}, i.e.,*

$$E[\bar{R}_{t+1} \mid x_{t+1}, \hat{c}_{t+1:T}] = \max_{c_{t+1:T}} E[\bar{R}_{t+1} \mid x_{t+1}, c_{t+1:T}] = V_{t+1}(x_{t+1}) \tag{12}$$

for $x_{t+1} = 1, \cdots, n_x$. Let $w_i \geq 0$ for $i = 1, \cdots n_x$. Then

$$\max_{c_{t+1:T}} \sum_{i=1}^{n_x} w_i E[\bar{R}_{t+1} \mid X_{t+1} = i, c_{t+1:T}] = \sum_{i=1}^{n_x} w_i E[\bar{R}_{t+1} \mid X_{t+1} = i, \hat{c}_{t+1:T}] \tag{13}$$

Proof. It follows from the *Max Sum* Lemma using

$$x \stackrel{\text{def}}{=} c_{t+1:T} \tag{14}$$

$$f_i(x) \stackrel{\text{def}}{=} E[\bar{R}_{t+1} \mid X_{t+1} = i, c_{t+1:T}] \tag{15}$$

□

Corollary 1.1 (**Bellman Equation for Optimal State/Value Function**).

$$\boxed{V_t(x_t, u_t) = E[R_t \mid x_t, u_t] + \gamma E[V(X_{t+1}) \mid x_t, u_t]} \tag{16}$$

Proof.

$$V_t(x_t, u_t) \stackrel{\text{def}}{=} \max_{c_{t+1:T}} E[\bar{R}_t \mid x_t, u_t, c_{t+1:T}] \tag{17}$$

$$= E[R_t \mid x_t, u_t] + \max_{c_{t+1:T}} E[\bar{R}_{t+1} \mid x_t, u_t, c_{t+1:T}] \tag{18}$$

where we used the fact that $\bar{R}_t = R_t + \gamma \bar{R}_{t+1}$ and the fact that R_t is conditionally independent of $c_{t+1:T}$ given x_t, u_t. Now note

$$E[\bar{R}_{t+1} \mid x_t, u_t, c_{t+1:T}] = \sum_{i=1}^{n_x} p(X_{t+1} = i \mid x_t, u_t) E[\bar{R}_{t+1} \mid X_{t+1} = i, c_{t+1:T}] \tag{19}$$

where we used the facts that

$$p(X_{t+1} = i \mid x_t, u_t, c_{t+1:T}) = p(X_{t+1} = i \mid x_t, u_t) \tag{20}$$

and the fact that

$$E[\bar{R}_{t+1} \mid X_{t+1} = i, u_t, c_{t+1:T}] = E[\bar{R}_{t+1} \mid X_{t+1} = i, c_{t+1:T}] \tag{21}$$

Thus, using the *Optimality Theorem* with $w_i = p(X_{t+1} = i \mid x_t, u_t)$ it follows that

$$\max_{c_{t+1:T}} E[\bar{R}_{t+1} \mid x_t, u_t, c_{t+1:T}] = \sum_{i=1}^{n_x} p(X_{t+1} = i \mid x_t, u_t) E[\bar{R}_{t+1} \mid X_{t+1} = i, \hat{c}_{t+1:T}]$$

$$= \sum_{x_{t+1}=1}^{n_x} p(x_{t+1} \mid x_t, u_t) V_{t+1}(x_{t+1}) = E[V(X_{t+1}) \mid x_t, u_t] \tag{22}$$

□

Corollary 1.2 (**Bellman Equation for Optimal Value Function**).

$$\boxed{V_t(x_t) = \max_{u_t} V_t(x_t, u_t)} \tag{23}$$

Proof.

$$V_t(x_t) \stackrel{\text{def}}{=} \max_{c_t} \max_{c_{t:T}} E[\bar{R}_t \mid x_t, c_t, c_{t+1:T}] \tag{24}$$

$$= \max_{u_t} E[\bar{R}_t \mid x_t, u_t, \hat{c}_{t+1:T}] = \max_{u_t} V_t(x_t, u_t) \tag{25}$$

where we used the fact that when x_t is fixed, the controller c_t determines u_t, thus optimizing with respect to $c_t(x_t)$ for a fixed x_t is the same as optimizing with respect to u_t. □

Corollary 1.3 (**Bellman Equation for a Fiexed Controller**).

$$\boxed{V^c(x_t) = E[R_t + \gamma\, V^c_{t+1}(X_t) \mid x_t, u_t]} \tag{26}$$

Proof. First consider the case in which the admissible control laws are of the form

$$U_t = C_t(X_t) \in \mathcal{C}_t(X_t) \tag{27}$$

where $\mathcal{C}_t(x_t)$ is a set of available actions when visiting state x_t at time t. This can be seen as a special case of the optimal control problem that happens to have have a large negative constant added to the reward function when using inadmissible actions. Thus

$$V(x_t) = \max_{u_t \in \mathcal{C}_t(x_t)} E[R_t + \gamma\, V_{t+1}(X_t) \mid x_t, u_t] \tag{28}$$

To get the value of a fixed controller c simply restrict the set $\mathcal{C}_t(x_t) = \{c_t(x_t)\}$ and apply (28)

$$V^c_t(x_t) = E[R_t + \gamma V^c_{t+1}(X_{t+1}) \mid x_t, c] \tag{29}$$

□

Corollary 1.4 (**Optimal Controller**). *The optimal action at time t given that we are at state x_t is as follows*[1]

$$\boxed{\hat{c}_t(x_t) \operatorname*{argmax}_{u_t} Q_t(x_t, c_t)} \tag{30}$$

Proof.

$$\hat{c}_t(x_t) \stackrel{\text{def}}{=} \operatorname*{argmax}_{u_t} \max_{c_{t+1:T}} E[\bar{R}_t \mid x_t, c_{t:T}] = \operatorname*{argmax}_{c_t} Q_t(x_t, c_t) \tag{31}$$

□

Remark 1.2 (**Backpropagation Algorithm for MDPs**). This suggests a useful method for finding optimal controllers: First we get the Q_T, V_T functions and the optimal controller $\hat{c}_T$ for the terminal time T:

$$V_T(x_T, u_T) = E[R_T \mid x_T, u_T] \tag{32}$$

$$V_T(x_T) = \max_{u_T} Q_T(x_T, u_T) \tag{33}$$

$$\hat{c}_T(x_T) = \operatorname*{argmax}_{u_T} Q_T(x_T, u_T) \tag{34}$$

We can then use the Bellman Equations to find the optimal value functions and optimal controllers

$$V_{T_1}(x_{T-1}, u_{T-1}) = E[R_{T-1} \mid x_{T-1}, u_{T-1}] + \gamma \sum_{x_T} p(x_T \mid x_{T-1}, u_{T-1}) V_T(x_T) \tag{35}$$

$$V_{T-1}(x_{T-1}) = \max_{u_{T-1}} V_{T-1}(x_{T-1}, u_{T-1}) \tag{36}$$

$$\hat{c}_T(x_T) = \operatorname*{argmax}_{u_{T-1}} V_{T-1}(x_{T-1}, u_{T-1}) \tag{37}$$

$$\tag{38}$$

The process can be iterated backwards in time down to any desired time t to find the optimal control law $\hat{c}_{t:T}$

$$
\begin{aligned}
&v^u_T = r^u_T, \quad \text{for } u = 1, \cdots, n_u.\\
&v_{i,T} = \max_u v^u_{i,T}, \quad \text{for } u = 1, \cdots, n_u,\ i = 1, \cdots, n_x.\\
&v^u_t = r^u_t + a^u v_{t+1}, \quad \text{for } t = T-1, \cdots, 1,\ u = 1 \cdots, n_u.\\
&v_{i,t} = \max_u q^u_{i,t} \quad \text{for } t = T-1, \cdots, 1,\ u = 1 \cdots, n_u,\ i = 1, \cdots, n_x.
\end{aligned}
$$

Figure 2: *The backpropagation dynamic programing algorithm for computing the optimal value functions and optimal controller in MDPs. The r^u_t, v^u_t, v_t terms are n_x dimensional vectors.*

Remark 1.3 (**Assumptions).** It is useful to clarify the assumptions made to prove the optimality principle and the Bellman Optimality Equations.

- Assumption 1:
$$E[R_t \mid x_t, c_{t:T}] = E[R_t \mid x_t, c_t] \tag{39}$$
- Assumption 2:
$$p(x_{t+1} \mid x_t, c_t, c_{t+1:T}) = p(x_{t+1} \mid x_t, c_t) \tag{40}$$
- Assumption 3:
$$E[\bar{R}_{t+1} \mid x_t, c_t, x_{t+1}, c_{t+1:T}] = E[\bar{R}_{t+1} \mid x_{t+1}, c_{t+1:T}] \tag{41}$$
- Assumption 4: Most importantly we assumed that the optimal controller $\hat{c}_{t+1:T}$ did not impose any constraints on the set of policies c_t with respect to which we were performing the optimization. This would be violated, if there were an additional penalty or reward that depended directly on $c_{t:T}$. For example, this assumption would be violated if we were to force the policies of interest to be stationary. This would amount to putting a large penalty for policies that do not satisfy $c_1 = c_2 = \cdots c_{T-1}$.

Remark 1.4 (Transition Dependent Rewards)**.** In some problems the reward R_t may be a function of X_t, X_{t+1}, U_t. Note this does not brake any of the assumptions and so the Bellman equations hold. In cases like this it would probably be a good idea to change the notation so R_t would be referred to as R_{t+1} to reflect the dependency on variable that is generated at time $t+1$.

2 Partially Observable Processes: Finite Horizon

In addition to the processes defined in the fully observable case we have the following additional processes

- *Sensor Process*: $Z = \{Z_1, \cdots, Z_T\}$. Where the sensor measurements Z_t take values in $\{1, \cdots, n_z\}$
- *Observation History*: $H_t = (Z_{1:t}, U_{1:t-1})$

[1] We use the term $\operatorname{argmax}_x f(x)$ in an informal sense to signify any value that globally maximizes f, i.e., if $\hat{x} = \operatorname{argmax} f(x)$ then $f(\hat{x}) = \max f(x)$.

- *Control Process*: $C = \{C_1, \cdots, C_t\}$. In this case the controller does not have access to the system variables. Instead at each time step t the controller $C_t : H_t \to U_t$ maps the available information into actions.
- Generative Model (See Figure 6).

  (H_t, C_t) generate U_t.
  (X_t, U_t) generate (R_t, X_{t+1}, Z_{t+1}).
  (H_t, U_t, Z_{t+1}) generate H_{t+1}.
- *Sensor matrices:* $b^u = \{b^1, \cdots, b^{n_u}\}$, where b^u is an $n_x \times n_z$ matrix with

$$b^u_{i,j} = p(Z_t = j \mid X_t = i, U_t = u) \tag{42}$$

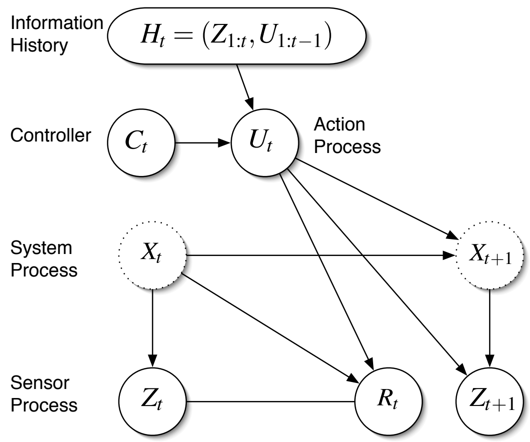


Figure 3: *Graphical Representation of the a time slice of the process. Arrows represent dependency relationships between variables. An arrow from variable X to variable Y indicates that X is a* parent *of Y. The probability of a random variable is conditionally independent of all the other variables given the parent variables. Dotted figures indicate unobservable variables, continuous figures indicate observable variables.*

2.1 Equivalence with Fully Observable Case

The goal in a finite horizon POMPD problem is to find control policies c that map the observable history into actions in an optimal manner. Here optimality is defined with respect to a reward process $R_t(X_t, U_t)$

$$V^c_t(h_t) = E[\bar{R}_t \mid h_t, c] \tag{43}$$

It turns this optimization problem is an MDP problem with respect to the information state. Note H_t satisfies the necessary assumptions

- Assumption 1:
$$E[R_t \mid h_t, c_{t:T}] = E[R_t \mid h_t, c_t] \tag{44}$$
- Assumption 2:
$$p(o_{t+1} \mid h_t, c_t, c_{t+1:T}) = p(o_{t+1} \mid h_t, c_t) \tag{45}$$
- Assumption 3:
$$E[\bar{R}_{t+1} \mid h_t, c_t, h_{t+1}, c_{t+1:T}] = E[\bar{R}_{t+1} \mid h_{t+1}, c_{t+1:T}] \tag{46}$$

2.2 Sufficient Statistics

While it is useful to know that POMPDs are MDPs with respect to the information state, the problem is that the number of information grows exponentially with the horizon T. For example if at each time we can get a binary sensory value Z_t and a binary actuator value U_t then by time T there is total of 4^T possible states. Fortunately in some cases one may find a sufficient statistic S_t of H_t that does not loose the relevant information about H_t. To be usable, such a statistic would need to satisfy the conditions of the following theorem.

Theorem 2.1 (Sufficient Statistics). *Let S_t be a random variable satisfying the following conditions:*

- *Assumption 1: It is a function of H_t*
$$S_t = f_t^1(H_t) \tag{47}$$
- *Assumption 2: It is recursive:*
$$S_{t+1} = f_t^2(S_t, U_t, Z_{t+1}) \tag{48}$$
- *Assumption 3: The expected reward given h_t, u_t is function of s_t, u_t*
$$E[R_t \mid h_t, u_t] = f_t^3(s_t, u_t) \tag{49}$$
where $s_t \stackrel{def}{=} f_t^1(h_t)$
- *Assumption 4: The distribution of sensor measurements given h_t, u_t is a function of s_t, u_t*
$$p(z_{t+1} \mid h_t, u_t) = f_t^4(z_{t+1}, s_t, u_t) \tag{50}$$
where $s_t \stackrel{def}{=} f_t^1(h_t)$

Then the optimal controller given H_t is the same as the optimal controller given S_t.

Proof. First we show that at terminal time T we can recover $V_T(h_t, u_t)$ using a function $\tilde{V}_T$ of s_t, u_t. For a fixed h_T let $s_T \stackrel{\text{def}}{=} f_T^1(h_T)$ and note
$$V_T(h_T, u_t) = E[R_t \mid h_T, u_T] = f_T^3(s_T, u_T) \tag{51}$$
Let
$$\tilde{V}_T(s_T, u_t) \stackrel{\text{def}}{=} f_T^3(s_T, u_T) = V_T(h_T, u_t) \tag{52}$$
The same argument can be used to show that $V_T(h_t)$ and the optimal action $\hat{c}_t(h_t)$ are functions of s_t
$$\tilde{V}_T(s_t) \stackrel{\text{def}}{=} \max_{u_T} \tilde{V}_T(s_t, u_t) = V_T(h_T) \tag{53}$$
$$\tilde{c}_T(s_t) \stackrel{\text{def}}{=} \hat{c}_T(h_t) = \operatorname*{argmax}_{u_T} \tilde{V}_T(s_t, u_t) \tag{54}$$
Now assume for time $t+1$, the optimal value of an information state h_t can be recovered from the statistic $s_{t+1} \stackrel{\text{def}}{=} f^1 t + 1(h_{t+1})$, i.e., there is a function $\tilde{V}_{t+1}$ such that
$$\tilde{V}_{t+1}(s_{t+1}) \stackrel{\text{def}}{=} V_{t+1}(h_{t+1}) \tag{55}$$
We will now show that if this assumption holds, then the optimal value and the optimal actions at time t for every information state h_t can be computed from their

sufficient statistic $s_t \stackrel{\text{def}}{=} f_t^1(h_t)$. Let h_t be an arbitrary sample of H_t and s_t its sufficient statistic, i.e., $s_t = f_t^1(h_t)$. Using Bellman's equation we get

$$V_t(h_t, u_t) = E[R_t \mid h_t, u_t] + \gamma \sum_{h_{t+1}} p(h_{t+1} \mid h_t, u_t) V_{t+1}(h_{t+1}) \tag{56}$$

where

$$\begin{aligned} p(h_{t+1} \mid h_t, u_t) &= \sum_{z_{t+1}} p(h_{t+1}, z_{t+1} \mid h_t, u_t, z_{t+1}) \\ &= \sum_{z_{t+1}} p(z_{t=1} \mid h_t, u_t) p(h_{t+1} \mid h_t, u_t) \end{aligned} \tag{57}$$

Note h_t, u_t, z_{t+1} determine the information history at time $t+1$, i.e.

$$p(h_{t+1} \mid h_t, u_t, z_{t+1}) = \delta(h_{t+1}, f_t^1(h_t, u_t, z_{t+1})) \tag{58}$$

$$p(h_{t+1} \mid h_t, u_t) = \sum_{z_{t+1}} p(z_{t+1} \mid h_t, u_t) \delta(h_{t+1}, f_t^1(h_t, u_t, z_{t+1})) \tag{59}$$

Thus

$$V_t(h_t, u_t) = E[R_t \mid h_t, u_t] + \gamma \sum_{z_{t+1}} p(z_{t+1} \mid h_t, u_t) V_{t+1}(f_t^1(h_t, u_t, z_{t+1})) \tag{60}$$

$$= E[R_t \mid s_t, u_t] + \gamma \sum_{z_{t+1}} p(z_{t+1} \mid s_t, u_t) \tilde{V}_{t+1}(f_t^2(s_t, u_t, z_t)) \tag{61}$$

which is a function of s_t, u_t, i.e.

$$\tilde{V}_t(s_t, u_t) \stackrel{\text{def}}{=} V_t(h_t, u_t) = E[R_t \mid s_t, u_t] + \gamma \sum_{z_{t+1}} p(z_{t+1} \mid s_t, u_t) \tilde{V}_{t+1}(f_t^2(s_t, u_t, z_t)) \tag{62}$$

and

$$\tilde{V}_t(s_t) \stackrel{\text{def}}{=} V_t(h_t) = \max_{u_t} \tilde{Q}_t(s_t, u_t) \tag{63}$$

$$\tilde{c}_t(s_t) \stackrel{\text{def}}{=} \hat{c}_t(h_t) = \max_{u_t} \tilde{Q}_t(s_t, u_t) \tag{64}$$

$$\tag{65}$$

Thus, starting at T and moving backwards in time, we can compute all the necessary value functions and optimal actions for each possible information state h_t using just the sufficient statistic of h_t. □

Theorem 2.2 (Controller Given the Posterior State Distribution). *The optimal controller given the information history h_t is equivalent to the optimal controller given the posterior distribution $p(x_t \mid h_t)$.*

Proof. We just need to show that the posterior distribution satisfies the assumption of the Sufficient Statistics Theorem.

- The posterior distribution is a function of the information history. This is obviously true since

$$p(x_t \mid h_t) = f_t^1(h_t) \tag{66}$$

- The posterior distribution is a recursive function. For a fixed h_t, u_t, z_{t+1} let $h_{t+1} \stackrel{\text{def}}{=} f_t^2(h_t, u_t, z_{t+1})$. Thus

$$p(x_{t+1} \mid h_{t+1}) = p(x_{t+1} \mid h_t, u_t, z_{t+1}) = \frac{p(x_{t+1}, z_{t+1} \mid h_t, u_t)}{p(z_{t+1} \mid h_t, u_t)} \tag{67}$$

where

$$\begin{aligned} p(x_{t+1}, z_{t+1} \mid h_t, u_t) &= \sum_{x_t} p(x_t, x_{t+1}, z_{t+1} \mid h_t, u_t) \\ &= \sum_{x_t} p(x_t \mid h_t) p(x_{t+1} \mid x_t, u_t) p(z_{t+1} \mid x_{t+1}, u_t) \\ &= \sum_{x_t} p(x_t \mid h_t) a^{u_t}_{x_t, x_{t+1}} b^{u_t}_{x_{t+1}, z_{t+1}} \end{aligned} \tag{68}$$

and

$$\begin{aligned} p(z_{t+1} \mid h_t, u_t) &= \sum_{x_{t+1}} p(x_{t+1}, z_{t+1} \mid h_t, u_t) \\ &= \sum_{x_t} p(x_t \mid h_t) \sum_{x_{t+1}} a^{u_t}_{x_t, x_{t+1}} b^{u_t}_{x_{t+1}, z_{t+1}} \end{aligned} \tag{69}$$

Thus

$$p(x_{t+1} \mid h_{t+1}) = \frac{p(x_t \mid h_t) \sum_{x_{t+1}} a^{u_t}_{x_t, x_{t+1}} b^{u_t}_{x_{t+1}, z_{t+1}}}{\sum_{x_t} p(x_t \mid h_t) \sum_{x_{t+1}} a^{u_t}_{x_t, x_{t+1}} b^{u_t}_{x_{t+1}, z_{t+1}}} \tag{70}$$

is a function of $p(x_t \mid h_t), u_t, z_{t+1}$

- The expected reward is a function of the posterior distribution. This is obviously true since

$$E[R_t \mid h_t, u_t] = \sum p(x_t \mid h_t) R(x_t, u_t) \tag{71}$$

- The distribution of sensor measurements is a function of the posterior distributions. This is evident in equation (69).

□

2.3 Backpropagation Algorithm for POMDPs

We will follow the convention that X_t, U_t cause the state X_{t+1}. The state X_{t+1} and, possibly, the action U_t cause the observation Z_{t+1}. Let n_x, n_u, n_z be the number of possible states, actions, and observations. Let H_t be the observable history up to time t, i.e,

$$H_t = (Z_{1:t}, U_{1:t-1}) \tag{72}$$

and Q_t represent the posterior probability of the states given the observations available up to that time, i.e.,

$$q_{i,t} = p(X_t = i \mid h_t) \tag{73}$$

Hereafter we refer to Q_t as the *information state* at time t. Given an information state q_t our goal is to find a controller that maps information states for all time steps q into actions so as to optimize the accumulation of reward over a finite period of time. The value of an information state q_t given a controller c, is defined as follows

$$V_t^c(q_t) = E[\bar{R}_t \mid q_t, c] \tag{74}$$

where

$$\bar{R}_t \stackrel{\text{def}}{=} \sum_{\tau=t}^{T} \gamma^{\tau-t} R_\tau \tag{75}$$

and R_t is a function of X_t, U_t. The case in which R_t is also a function of X_{t-1} can be handled as described in Remark 1.4. Let

$$a_{ij}^u \stackrel{\text{def}}{=} p(X_{t+1} = j \mid X_t = i, U_t = u) \tag{76}$$

$$b_{ij}^u \stackrel{\text{def}}{=} p(Z_{t+1} = j \mid X_{t+1} = i, U_t = u) \tag{77}$$

$$\tag{78}$$

Let $\mathbf{b}_z^u$ be a diagonal matrix whose i^{th} diagonal elements is $b_{i,z}^u$, i.e,

$$\mathbf{b}_z^u \stackrel{\text{def}}{=} \operatorname{diag}(b_{\cdot,z}^u) \tag{79}$$

Let

$$r_{i,t}^u = R_t(i, u) \tag{80}$$

Let

$$h_j(q_t, u, z) \stackrel{\text{def}}{=} p(X_{t+1} = j, Z_{t+1} = z \mid q_t, U_t = u) \tag{81}$$

Thus

$$h_j(q_t, u, z) = \sum_{i=1}^{n_x} q_{i,t}\; a_{i,j}^u\; b_{j,z}^u \tag{82}$$

or in vector form

$$h(q, u, z) = q' a^u \mathbf{b}_z^u \tag{83}$$

Let

$$f_j(q_t, u, z) \stackrel{\text{def}}{=} p(X_{t+1} = j \mid q_t, U_t = u, Z_{t+1} = z) \tag{84}$$

Thus

$$f_j(q_t, u, z) = \frac{h_j(q_t, u, z)}{p(Z_{t+1} = z \mid q_t, U_t = u)} \tag{85}$$

where

$$p(Z_{t+1} = z \mid q_t, U_t = u) = \sum_{j=1}^{n_x} h_j(q_t, u, z) \tag{86}$$

We now note that the transition probability between information states is a sum of delta functions

$$p(q_{t+1} \mid q_t, u_t) = \sum_{z_{t+1}} p(z_{t+1} \mid q_t, u_t)\delta(q_{t+1}, f(q_t, u_t, z_{t+1})) \tag{87}$$

where δ is the Kronecker delta function

$$\delta(u,v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{else} \end{cases} \tag{88}$$

and

$$f(q,u,z) = \Big(f_1(q,u,z), \cdots, f_{n_x}(q,u,z)\Big)' \tag{89}$$

Let $V_t(q_t, u_t)$ the optimal value of the belief q_t and action u_t at time t, i.e.,

$$V_t(q_t, u_t) = q' r_t^u + \gamma \sum_{q_{t+1}} p(q_{t+1} \mid q_t, u_t) V_{t+1}(q_{t+1}) \tag{90}$$

where V_{t+1} is the optimal value function at time $t+1$, and γ is the discount factor. Thus

$$\begin{aligned} V_t(q_t, u_t) &= q' r_t^u + \gamma \sum_{q_{t+1}} \sum_{z_{t+1}} p(z_{t+1} \mid q_t, u_t) \delta(q_{t+1}, f(q_t, u_t, z)) V_{t+1}(q_{t+1}) \\ &= q' r_t^u + \gamma \sum_{z_{t+1}} p(z_{t+1} \mid q_t, u_t)\ V_{t+1}(f(q_t, u_t, z_{t+1})) \end{aligned} \tag{91}$$

We will now assume (and later prove) that there is a matrix w_{t+1} such that

$$V_{t+1}(q) = \max(q' w_{t+1}) \tag{92}$$

Let m_{t+1} be the number of columns of w_{t+1}. Note $q' w_{t+1}$ is an m_{t+1} dimensional row vector. The max operator simply chooses an element of this vector that is not smaller than any other element of the vector.

In such case, for all $\alpha \in \Re$

$$V_{t+1}(\alpha q) = \alpha V_{t+1}(q) \tag{93}$$

Thus

$$V_t(q_t, u_t) = q' r_t^u + \gamma \sum_{z_{t+1}} V_{t+1}(p(z_{t+1} \mid q_t, u_t) f(q_t, u_t, z_{t+1})) \tag{94}$$

$$= q' r_t^u + \gamma \sum_{z=1}^{n_z} V_{t+1}(h(q_t, u_t, z)) \tag{95}$$

which under assumption (92) simplifies as follows

$$V_t(q_t, u_t) = q' r_t^u + \gamma \sum_{z=1}^{n_z} \max(h(q_t, u_t, z)) w) \tag{96}$$

$$= q' r_t^u + \gamma \sum_{z=1}^{n_z} \max(q' a^u \mathbf{b}_z^u w_{t+1}) \tag{97}$$

and using Lemma 4.2 (sum of max is max of cross sums)

$$V_t(q,u) = q' r_t^u + \gamma \max\Big(q' \bigoplus_{z=1}^{n_z} a^u \mathbf{b}_z^u w_{t+1}\Big) \tag{98}$$

$$= \max(q' w_t^u) \tag{99}$$

where

$$w_t^u \stackrel{\text{def}}{=} \Big(\bigoplus_{z=1}^{n_z} \gamma a^u \mathbf{b}_z^u w_{t+1}\Big) \oplus r_t^u \tag{100}$$

$$
\begin{aligned}
&w_{T+1} = (0, \cdots, 0)' \in \Re^{n_x} \\
&\text{For } t = T, T-1 \cdots, 1 \\
&\quad k_t^{u,z} \stackrel{\text{def}}{=} a^u \text{diag}(b_{\cdot,z}), \quad \text{for } u = 1, \cdots n_u \text{ and } z = 1 \cdots n_z \\
&\quad w_t^{u,z} \stackrel{\text{def}}{=} \gamma k_t^{u,z}\; w_{t+1}, \quad \text{for } u = 1, \cdots n_u \text{ and } z = 1 \cdots n_z \\
&\quad w_t^u \stackrel{\text{def}}{=} r_t^u \oplus w_t^{u,1} \oplus \cdots \oplus w_t^{u,n_z}, \quad \text{for } u = 1, \cdots n_u \\
&\quad w_t = \left(w_t^1, \cdots w_t^{n_u} \right) \\
&\quad V_t(q, u) = \max_{cols} q' w_t^u, \quad \text{for } u = 1, \cdots n_u \\
&\quad V_t(q) = \max_u V_t(q, u);
\end{aligned}
$$

Figure 4: *The backpropagation dynamic programing algorithm for computing the optimal value functions and optimal controller in POMDPs.*

Note the *cross sum* operator $\oplus$ and its properties are described in the Appendix. Thus

$$V_t(q) = \max_u V_t(q, u) = \max_u \max(q' w_t^u) = \max(q' w_t) \tag{101}$$

where w_t is a column-wise concatenation of the w_t^u matrices, i.e.,

$$w_t \stackrel{\text{def}}{=} \left(w_t^1 w_t^2 \cdots w_t^{n_u} \right) \tag{102}$$

Thus we have shown that if assumption (92) is true for time $t+1$ then it must also be true for time t. In addition equations (100) and (102) tell us how to construct the weight matrix w_t given the weight matrix w_{t+1}. Now note that for $t = T$

$$V_T(q, u) = q' r_T^u \tag{103}$$

$$V_T(q) = \max_u V_T(q, u) = \max(q' w_T) \tag{104}$$

$$w_T \stackrel{\text{def}}{=} \left(r_T^1 \cdots r_T^{n_u} \right) \tag{105}$$

thus this shows, by induction, that assumption (92) is correct and provides a method to compute the value function starting at T and going all the way back to t.

Note for time T $w_T = (r_T^1 \cdots r_T^{n_u})$ has n_u columns. For time t we have n_u w_t^u matrices. For each w_t^u we cross sum n_z matrices each with as many columns as the number of columns in w_t this gives a total of $n_{t+1}^{n_z}$ columns, where n_{t+1} represents the number of columns in w_{t+1}. For each w_t^u we cross sum r_t^u, which is a vector so it does not increase the number of columns. w_t concatenates the w_t^u matrices. Thus the w_t^u matrix has $n_t = (n_{t+1}^{n_z}) * n_u$. Suppose $n_u = 2, n_z = 2$, then $n_T = 2$, $n_{T-1} = (2^2) * 2 = 8$, $n_{T-2} = (8^2) * 2 = 128$, etc.

3 Infomax Control

In some problems of interest the goal is to act in a manner that provides the most information about the state of the world. In such cases it is useful to use the entropy of the posterior distribution as a component of the reward function. The Renyi entropy of order $\alpha \geq 0$ is defined as follows (I need to check the paper Blind Source Separation Using Renyi's mutual information)

$$H_\alpha(p) = \frac{1}{1-\alpha} \log(\sum_x p^\alpha(x)) \tag{106}$$

It can be shown that in the limit as $\alpha \to 1$ H_α converges to the Shannon Entropy

$$\lim_{\alpha \to 1} H_\alpha(p) = \sum_x p(x) \log p(x) \tag{107}$$

Moreover

$$H_\infty(p) = \lim_{\alpha \to \infty} H_\alpha(X) = -\log\left(\max_x p(x)\right) \tag{108}$$

We can adapt the backpropagation algorithm to solve infomax problems by using

$$\exp(-H_\infty(q_t)) = \max_x q_t(x) \tag{109}$$

as a component of the reward function. To this end we let the instantaneous reward associated with action u when in information state q is a follows

$$\left(\sum_x q_t R_t(x,u)\right) + \lambda \max_x q_t(x) \tag{110}$$

for a parameter $\lambda \geq 0$ that controls the relative importance of the entropy term. In this case (91) takes teh following form

$$V_t(q_t, u_t) = q' r_t^u + \max \lambda q' + \gamma \sum_{z_{t+1}} p(z_{t+1} \mid q_t, u_t)\; V_{t+1}(f(q_t, u_t, z_{t+1})) \tag{111}$$

We assume (and later prove) that there is a matrix w_{t+1} such that

$$V_{t+1}(q) = \max(q' w_{t+1}) \tag{112}$$

Following the same steps as in (94) to (98) we get

$$V_t(q,u) = q' r_t^u + \; \max q' \lambda I + \gamma \max(q' \bigoplus_{z=1}^{n_z} a^u \mathbf{b}_z^u w_{t+1}) = \max(q' w_t^u) \tag{113}$$

where I is an $n_x \times n_x$ identity matrix and

$$w_t^u = \left(\bigoplus_{z=1}^{n_z} \gamma a^u \mathbf{b}_z^u w_{t+1}\right) \oplus r_t^u \oplus \lambda I \tag{114}$$

Thus

$$V_t(q) = \max_u V_t(q,u) = \max_u \max(q' w_t^u) = \max(q' w_t) \tag{115}$$

where w_t is the column-wise concatenation of the w_t^u matrices

$$w_t = (w_t^1 \cdots w_t^{n_u}) \tag{116}$$

All is left is to show that assumption (92) is true for time T. Note

$$V_T(q,u) = q' r_T^u + \lambda \max(q') \tag{117}$$

Thus

$$V_T(q) = \max(q' r_t) + \max(q' \lambda I) = \max q' w_T \tag{118}$$

$$
\begin{aligned}
& w_{T+1} = (0, \cdots, 0)' \in \Re^{n_x} \\
& \text{For } t = T, T-1 \cdots, 1 \\
& \quad k_t^{u,z} \stackrel{\text{def}}{=} a^u \text{diag}(b_{\cdot,z}), \;\; \text{for } u = 1, \cdots n_u \text{ and } z = 1 \cdots n_z \\
& \quad w_t^{u,z} \stackrel{\text{def}}{=} \gamma k_t^{u,z} \; w_{t+1}, \;\; \text{for } u = 1, \cdots n_u \text{ and } z = 1 \cdots n_z \\
& \quad w_t^u \stackrel{\text{def}}{=} \lambda I \oplus r_t^u \oplus w_t^{u,1} \oplus \cdots \oplus w_t^{u,n_z}, \;\; \text{for } u = 1, \cdots n_u \\
& \quad w_t = \left(w_t^1, \cdots w_t^{n_u} \right) \\
& \quad V_t(q,u) = \max_{cols} q' w_t^u, \;\; \text{for } u = 1, \cdots n_u \\
& \quad V_t(q) = \max_u V_t(q,u);
\end{aligned}
$$

Figure 5: *The backpropagation dynamic programing algorithm for Infomax Control in POMDPs.*

where

$$w_T = r_T \oplus \lambda I \tag{119}$$

$$r_T = (r_T^1 \cdots r_T^{n_u}) \tag{120}$$

Note for time T we have n_u matrices w_T^u. Each $w_T^u = r_T^u \oplus I_{n_x}$ has n_x columns. Thus w_T has $n_u \times n_x$ columns. For time t we have n_u matrices w_t^u. For each w_t^u we cross sum n_z matrices each with as many columns as the number of columns in w_t this gives a total of $n_{t+1}^{n_z}$ columns, where n_{t+1} represents the number of columns in w_{t+1}. For each w_t^u we cross the I_{n_x} thus the number of columns for each is $(n_{t+1}^{n_z})n_x$. For each w_t^u we cross sum r_t^u, which is a vector so it does not increase the number of columns. w_t concatenates the w_t^u matrices.

Thus the w_t^u matrix has $n_t = (n_{t+1}^{n_z})n_x n_u$. Suppose $n_u = 2, n_z = 2$, then $n_T = 4$, $n_{T-1} = (4^2) * 4 = 64$, $n_{T-2} = (64^2) * 4 = 16384$, etc.

3.1 A Counterintuitive Example

Consider the following case: There are two internal states, two actions and two observations. The state transition probability is the identity matrix

$$a^u = I \;\; \text{for } u = 1, 2 \tag{121}$$

The first action provides no information about the state

$$b_{\cdot z}^1 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \;\; \text{for } z = 1, 2 \tag{122}$$

where I is the identity matrix. The second action provides information about the state

$$b_{\cdot 1}^2 = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} \tag{123}$$

$$b_{\cdot 2}^2 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \tag{124}$$

If we choose action 2 and we get observation 1, then this indicates that the system is likely to be in state 1. If we choose action 2 and get observation 2, then it is likely to be in state 2. At time t the controller has a belief state q_t then it chooses one of the two actions. The total reward is the max value of the components of q_t plus the max value of the components of q_{t+1}. This is related to the limit Renyii entropy with parameter α
$to\infty$. It also corresponds to the the probability of correctly guessing the state at time t plus the probability of correctly guessing the state at time $t+1$. Using (**??**) we get

$$V_1(q_t, u) = \max q_t + \sum_{z_{t+1}} p(z_{t+1} \mid q_t, u_t) \max_{x_{t+1}} p(x_{t+1} \mid q_t, z_{t+1}) \tag{125}$$

$$= \max q_t + \sum_{z_{t+1}} \max_{x_{t+1}} p(z_{t+1} \mid q_t, u_t)\, p(x_{t+1} \mid q_t, z_{t+1}) \tag{126}$$

$$= \max q_t + \sum_{z_{t+1}} \max_{x_{t+1}} p(x_{t+1}, z_{t+1} \mid q_t, u_t) \tag{127}$$

Note for $U_t = 1$

$$p(Z_{t+1} = 1, X_{t+1} = 1 \mid q_t, U_t = 1) = q_{1t}a_{11}b^1_{11} + q_{2t}a_{21}b^1_{11} = q_{t1}0.5 \tag{128}$$

$$p(Z_{t+1} = 2, X_{t+1} = 1 \mid q_t, U_t = 1) = q_{1t}a_{11}b^1_{12} + q_{2t}a_{21}b^1_{12} = q_{1t}0.5 \tag{129}$$

$$p(Z_{t+1} = 1, X_{t+1} = 2 \mid q_t, U_t = 1) = q_{1t}a_{12}b^1_{21} + q_{2t}a_{22}b^1_{22} = q_{2t}0.5 \tag{130}$$

$$p(Z_{t+1} = 2, X_{t+1} = 2 \mid q_t, U_t = 1) = q_{1t}a_{12}b^1_{22} + q_{2t}a_{22}b^1_{22} = q_{2t}0.5 \tag{131}$$

Thus

$$V(q_t, 1) = (\max q_t) + 0.5\ \max\{q_{1t}, q_{2t}\} + 0.5\ \max\{q_{1t}, q_{2t}\} = 2 \max q_t \tag{132}$$

Thus if we choose the uninformative action, the probability of being correct at time t equals the probability of being correct at time $t+1$. The total reward is simply twice the probability of being correct given the prior belief q_t. For $U_t = 2$

$$p(Z_{t+1} = 1, X_{t+1} = 1 \mid q_t, U_t = 2) = q_{1t}a_{11}b^2_{11} + q_{2t}a_{21}b^2_{11} = q_{1t}0.9 \tag{133}$$

$$p(Z_{t+1} = 2, X_{t+1} = 1 \mid q_t, U_t = 2) = q_{1t}a_{11}b^2_{12} + q_{2t}a_{21}b^2_{12} = q_{1t}0.1 \tag{134}$$

$$p(Z_{t+1} = 1, X_{t+1} = 2 \mid q_t, U_t = 2) = q_{1t}a_{12}b^2_{21} + q_{2t}a_{22}b^2_{21} = q_{2t}0.1 \tag{135}$$

$$p(Z_{t+1} = 2, X_{t+1} = 2 \mid q_t, U_t = 2) = q_{1t}a_{12}b^2_{22} + q_{2t}a_{22}b^2_{22} = q_{2t}0.9 \tag{136}$$

Thus

$$V(q_t, 1) = \max q_t + \max\{q_{1t}0.9, q_{2t}0.1\} + 0.5\ \max\{q_{1t}0.1, q_{2t}0.9\} \tag{137}$$

Consider the case in which we start with uninformative priors i.e., $q_t = (0.5, 0.5)'$. In this case

$$V(q_t, 1) = 2 \times 0.5 = 1 \tag{138}$$

$$V(q_t, 2) = 0.5 + 0.5 \max\{0.9, 0.1\} + 0.5 \max\{0.1, 0.9\} \tag{139}$$

$$= 0.5 + 0.9 = 1.4 > V(q_t, 1) \tag{140}$$

Thus, not surprisingly, in this case the optimal strategy is to choose the most informative action. Consider now the case for which $q_t = (0.9, 0.1)$. In this case

$$V(q_t, 1) = 2 \times 0.9 = 1.8 \tag{141}$$

$$V(q_t, 2) = 0.9 + \max\{0.81, 0.01\} + \max\{0.09, 0.09\} \tag{142}$$

$$= 0.90 + 0.81 + 0.09 = 1.8 = V(q_t, 1) \tag{143}$$

Thus, surprisingly, in this case it does not matter which action we take, the informative action is as good as the uninformative action.

3.2 Point Based Approximations

A problem with the algorithm described above is that the weight matrix w_t can grow very large. In particular if w_{t+1} has m_{t+1} rows, then w_t will have $n_u \times m_{t+1}^{n_z}$. In practice many of these columns may be irrelevant since they may never be picked by the max operator. Note if a column of w_t is never picked up by the max operator, it may be pruned out without any loss of information thus potentially mitigating the growth of w_t. In addition, a common approach is to focus on a finite set of information states q rather than in all possible states. Such an approach goes by the name of point based POMDP approximations.

3.3 Policy Gradient Methods

In policy gradient methods the controller is parameterized and we attempt to find values of the parameter that optimize a utility function. Here we will focus on the finite horizon case but the approach can be easily generalized for infinite horizon problems. Let θ represent the parameters of a controller i,e. for each θ there is a probability distribution that maps belief states into actions. Let

$$\Phi(\theta) = E[r(Q, Z, U)] \tag{144}$$

where $Q = Q_{1:T}$ is the belief process is determined by the observation process $Z = Z_{1:T}$ and the action process $U = U_{1:T}$. Note we let the reward r depend on the observable processes in a manner that may or may not be additive across time steps. Our goal is to find values of θ that maximize Φ. Note

$$\Phi(\theta) = \int p(q, z, u) r(q, z, u) \; dq \; dz \; du \tag{145}$$

where $q = q_{1:T}$, $z = z_{1:T}, u = u_{1:T}$. We will employ stochastic gradient descent methods to maximize Φ. Note

$$\begin{aligned} \nabla_\theta \Phi &= \int r(q, z, u) \; \nabla_\theta p(q, z, u) \; dq \; dz \; du \\ &= \int r(q, z, u) \, p(q, z, u) \; \nabla_\theta \log p(q, z, u) \; dq \; dz \; du \end{aligned} \tag{146}$$

Thus we can approximate the gradient by getting n samples from the observable processes $\{(q^{(i)}, z^{(i)}, u^{(i)}) \; : i = 1, \cdots n\}$

$$\nabla_\theta \Phi \approx \frac{1}{n} \sum_{i=1}^{n} r(q^{(i)}, z^{(i)}, u^{(i)}) \; \nabla_\theta \log p(q^{(i)}, z^{(i)}, u^{(i)}) \tag{147}$$

Note for any sequence (q, z, u) of belief, observations and actions

$$p(q, z, u \mid \theta) = p(q_0) \; p(z_1 \mid q_0) \; p(q_1 \mid q_0, z_1) \; p(u_1 \mid q_1, \theta) \tag{148}$$

$$p(z_2 \mid q_1, u_1) \; p(q_2 \mid q_1, u_1, z_2) \; p(u_2 \mid q_2, \theta) \cdots \tag{149}$$

Thus

$$\nabla_\theta \log p(q, z, u) = \sum_{t=1}^{T} \nabla_\theta \log p(u_t \mid q_t, \theta) \tag{150}$$

3.3.1 Soft-Max controller

A convenient way to parameterize the controller is to use a softmax function

$$p(U = i \mid q \; \theta) = \frac{e^{\theta'_{\cdot i} \phi(q)}}{\sum_k e^{\theta'_{\cdot k} \phi(q)}} \tag{151}$$

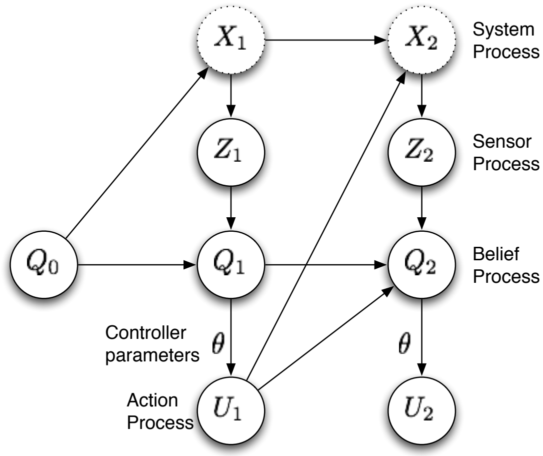


Figure 6: *Graphical Representation of a POMDP process. For policy gradient methods the controller is parameterize by θ and gradient methods are used to find values of θ that maximize the long term utility.*

where $\phi(q)$ is a vector function of q that represents key features of the belief state. Here we represent θ as a matrix. The element θ_{ij} of this matrix represents the support of feature $\phi_j(q)$ for response i. The term $\theta_{\cdot i}$ is the i column of the matrix θ. Let q be fixed and define

$$p_k = p(U = k \mid q\ \theta) \tag{152}$$

$$\phi = \phi(q) \tag{153}$$

Note

$$\frac{\partial p_k}{\partial \theta_{ij}} = p_k\ (\delta_{jk} - p_k)\ \phi_j \tag{154}$$

Thus

$$\frac{\partial \log p_k}{\partial \theta_{ij}} = (\delta_{jk} - p_k)\ \phi_j \tag{155}$$

4 Appendix

4.1 Terminology

Optimal controllers are presented in terms of maximization of a reward function. Equivalently they could be presented as minimization of costs, by simply setting the cost function equal the reward with opposite sign. Below is a list of useful words and their equivalents

- Cost = - Value = - Reward = - Utility.= - Payoff
- The goal is to minimize Costs, or equivalent to maximize Value, Reward, Utility.
- We will use the terms Return and Performance to signify Cost or Value.
- Step = Stage
- One Step Cost = Running Cost
- Terminal cost = Bequest cost
- Policy = control law = controller = control

- Optimal n-step to go cost = optimal 1 step cost + optimal (n-1) step to go cost
- n-step to go cost given policy = 1 step cost given policy + (n-1) step to go cost given policy

4.2 Cross Sums

Definition 4.1. *Let* $\mathbf{a}$ *be an* r *row,* c *column matrix and* $\mathbf{b}$ *an* r *row,* d *column matrix. Then* $\mathbf{a} \oplus \mathbf{b}$ *is an* r *row by* $c \times d$ *column matrix, defined as follows*

$$\mathbf{a} \oplus \mathbf{b} \stackrel{\text{def}}{=} \Big((\mathbf{a}_1 + \mathbf{b}_1), \cdots, (\mathbf{a}_1 + \mathbf{b}_d) + \cdots + (\mathbf{a}_c + \mathbf{b}_d) \Big) \tag{156}$$

where $\mathbf{a_i}, \mathbf{b}_j$ are the i^{th} column of $\mathbf{a}$ and j^{th} column of $\mathbf{b}$.

Definition 4.2. *Let* $\mathbf{a_1}, \cdots \mathbf{a_n}$ *be a set of matrices with* r *rows and* c_i *columns.*

$$\bigoplus_{i=1}^{n} \mathbf{a}_i \stackrel{def}{=} \mathbf{a}_1 \oplus \mathbf{a}_2 \oplus \cdots \oplus \mathbf{a}_n \tag{157}$$

Remark 4.1. Note $\bigoplus_{i=1}^{n} \mathbf{a}_i$ is a matrix with r rows and $\prod_{i=1}^{n} c_i$ columns

Lemma 4.1 (**Sum of Max is Max of Cross Sum).** *Let* x, y *be row vectors. Let* $\max(x)$ *be an element of* x *that is no smaller than any of the other elements of* x*. Then*

$$\max(x) + \max(y) = \max(x \oplus y) \tag{158}$$

Lemma 4.2. *Let* x *be an* r*-dimensional row vector. Let* $\mathbf{a}_1 \cdots \mathbf{a}_n$ *be matrices with* r *rows. Then*

$$\sum_{i=1}^{n} \max(x\ \mathbf{a}_i) = \max\Big(x\ \bigoplus_{i=1}^{n} \mathbf{a_i}\Big) \tag{159}$$