Minimum Jerk Trajectories

Neville Hollan showed that in many situations when we move our hands from one initial point to a target point, the trajectory minimizes the total jerk, i.e. ,the integral over the squared third derivative. Here we show the classical derivation of minimum jerk trajectories.

Let $x_0, x_0^{[1]}, x_0^{[2]}$ be the initial location, velocity and acceleration. Let T the terminal time, at which we want to achieve a target location, velocity and acceleration $x_T, x_T^{[1]}, x^{[2]}$. Our goal is to find a trajectory x that minimizes the integral of the squared jerk over time

$$I(x) = \frac{1}{2} \int_0^T (x_t^{[3]})^2 \; dt \tag{1}$$

where $x_t^{[3]}$ represents the third derivative of x_t with respect to time. For a fixed trajectory x let's define a family of functions of the following form

$$h(\epsilon, t) = x(t) + \epsilon \delta(t) \tag{2}$$

where δ is an arbitrary function with continuous second partial derivatives and such that $\delta_0 = \delta_T = 0$, $\delta_0^{[1]} = \delta_T^{[1]} = 0$, $\delta_0^{[2]} = \delta_T^{[2]} = 0$. Let

$$F(\epsilon) = \frac{1}{2} \int_a^b (h^{[3]})^2 \; dt \tag{3}$$

A necessary condition for the trajectory x to minimize I is that

$$\left. \frac{dF(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0 \tag{4}$$

Note

$$\frac{dF(\epsilon)}{d\epsilon} = \int_0^T (x_t^{[3]} + \epsilon \delta_t^{[3]}) \; \delta_t^{[3]} \; dt \tag{5}$$

and

$$\left. \frac{dF(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \int_0^T x_t^{[3]} \; \delta_t^{[3]} \; dt \tag{6}$$

Using integration by parts

$$\int_0^T x_t^{[3]}\ \delta_t^{[3]}\ dt = x_t^{[3]}\delta_t^{[2]}\Big|_0^T - \int_0^T x_t^{[4]}\ \delta_t^{[2]}\ dt \tag{7}$$

and since $\delta_0^{[2]} = \delta_T^{[2]} = 0$

$$\int_0^T x_t^{[3]}\ \delta_t^{[3]}\ dt = -\int_0^T x_t^{[4]}\ \delta_t^{[2]}\ dt \tag{8}$$

Using integration by parts again

$$\int_0^T x_t^{[4]}\ \delta_t^{[2]}\ dt = x_t^{[4]}\delta_t^{[1]}\Big|_0^T - \int_0^T x_t^{[5]}\ \delta_t^{[1]}\ dt \tag{9}$$

and since $\delta_0^{[1]} = \delta_T^{[1]} = 0$

$$\int_0^T x_t^{[4]}\ \delta_t^{[2]}\ dt = -\int_0^T x_t^{[5]}\ \delta_t^{[1]}\ dt \tag{10}$$

Using integration by parts one last time

$$\int_0^T x_t^{[5]}\ \delta_t^{[1]}\ dt = -\int_0^T x_t^{[6]}\ \delta_t\ dt \tag{11}$$

Thus

$$\frac{dF(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = 0 \tag{12}$$

requires that

$$\int_0^T x_t^{[6]}\ \delta_t\ dt = 0 \tag{13}$$

This must be the case for arbitrary functions δ, thus it must be the case that for all $t \in [0, T]$

$$x_t^{[6]} = 0 \tag{14}$$

Note a function any fifth order polynomial satisfies the constraint that the 6th derivative be zero everywhere, i.e.,

$$x_t = \sum_{k=0}^{5} a_k t^k \tag{15}$$

All that is needed now is to determine the six constants $a_0 \cdots a_5$ The first 3 constants can be determined from the initial conditions. For $t = 0$

$$a_0 = x_0 \tag{16}$$

$$a_1 = x_0^{[1]} \tag{17}$$

$$a_2 = \frac{1}{2} x_0^{[2]} \tag{18}$$

The last 3 constants can be determined from the terminal conditions

$$x_T = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4 + a_5 T^5 \tag{19}$$

$$x_T^{[1]} = a_1 + 2a_2 T + 3a_3 T^2 + 4a_4 T^3 + 5a_5 T^4 \tag{20}$$

$$x_T^{[2]} = 2a_2 + 6a_3 T + 12a_4 T^2 + 20a_5 T^3 \tag{21}$$

In matrix form

$$\begin{bmatrix} x_1 - a_0 - a_1 - a_2 \\ x_1^{[1]} - a_1 - 2a_2 \\ x_1^{[2]} - 2a_2 \end{bmatrix} = \begin{bmatrix} T^3 & T^4 & T^5 \\ 3T^2 & 4T^4 & 5T^5 \\ 6T & 12T^2 & 20T^3 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \\ a_5 \end{bmatrix} \tag{22}$$

Then

$$\begin{bmatrix} a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} T^3 & T^4 & T^5 \\ 3T^2 & 4T^4 & 5T^5 \\ 6T & 12T^2 & 20T^3 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - a_0 - a_1 - a_2 \\ x_1^{[1]} - a_1 - 2a_2 \\ x_1^{[2]} - 2a_2 \end{bmatrix} \tag{23}$$

Once the a parameters are known, the entire trajectory from start time 0 to terminal time T is deermined

$$x_t = \sum_{k=0}^{5} a_k t^k \tag{24}$$

1 On Line Version

In the previous section we precomputed a minimum jerk trajectory to get from an initial state to a final state in a desired time. We can also implement an online version of the minimum jerk algorithm that allows for the target states, and/or the target time, to change before the trajectory is completed. Given a current location, velocity and acceleration $x_t, x_t^{[1]}, x_t^{[2]}$ a reach time T and a target location, velocity, acceleration $x_T, x_T^{[1]}, x_T^{[2]}$. The location, velocity and acceleration of the minimum jerk trajectory at at time $t + \Delta_t$ can be obtained by getting the a parameters with starting point $x_t, x_t^{[1]}, x_t^{[2]}$, target point $x_t, x_t^{[1]}, x_t^{[2]}$ and reach time $T - t$. We can obtain the location, velocity and acceleration applying the following formulas

$$x_{t+\Delta t} = \sum_{k=0}^{5} a_k (\Delta t)^k \tag{25}$$

$$x_{t+\Delta t}^{[1]} = \sum_{k=1}^{5} k a_k (\Delta t)^{k-1} \tag{26}$$

$$x_{t+\Delta t}^{[2]} = \sum_{k=2}^{5} k(k-1) a_k (\Delta t)^{k-2} \tag{27}$$

We can then change the target states and target times, $t + \Delta t$ the new start time, get the minimum jerk a parameters and iterate.

2 Example Matlab Code

```
function a =  minimumJerk(x0, dx0, ddx0,xT,dxT,ddxT,T)
% Compute a point to point minimum jerk trajectory
% x0 dx0 ddx0 are the location, velocity and acceleration at the
% start point
% xT dxT ddxT are the target location velocity and acceleration
% T is the time required to move from the start point
% to the target point
%
% The solution is a 6-D vector of coefficients a
% The minimum jerk trajectory takes the form
% x_t = \sum_{k=1}^6 a_k t^(k-1), for 0\leq t \leq T
```

```
%
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T2 = T*T; T3 = T2*T;
T4 = T3*T; T5= T4*T;
a = zeros(6,1);
a(1) = x0;
a(2) = dx0;
a(3) = ddx0/2;
b= [T3 T4 T5 ; 3*T2 4*T3 5*T4; 6*T 12* T2 20* T3];
c = [ xT - a(1) - a(2)*T - a(3)*T2; dxT - a(2) - 2*a(3)*T;
      ddxT - 2*a(3)];
a(4:6,1)=pinv(b)*c;
```