Tutorial on Axiomatic Set Theory

Javier R. Movellan

Intuitively we think of sets as collections of elements. The crucial part of this intuitive concept is that we are willing to treat sets as entities distinguishable from their elements. Sometimes we identify sets by enumeration of their elements. For example, we may talk about the set whose elements are the numbers 1, 2 and 3. In mathematics such sets are commonly represented by embracing the elements of the set using curly brackets. For example, the set $\{1, 2, 3\}$ is the set whose elements are the numbers 1, 2 and 3. Sometimes sets are defined using some property that identifies their elements. In such case it is customary to represent the sets using the following formula

$$\{x : x \text{ has a particular property}\} \tag{1}$$

For example, the set $\{1, 2, 3\}$ can be represented as

$$\{x : x \text{ is a natural number smaller than } 4\}. \tag{2}$$

The intuitive concept of sets as collections of elements is useful but it can only take us so far. You may complain that we have not really defined what a set is since we have not defined collections and we have not specified what qualifies as an element. We have not specified either what qualifies as a property. Consider for example the proposition $\{x : x \text{ is not an element of } x\}$, i.e., the set of sets which are not elements of themselves. We can prove by contradiction that such a set does not exist. Let's assume that this set exists and lets represent it with the symbol y. If y is an element of y then, since all the elements of y are not an element of themselves it follows that y is not an element of y. Moreover, if y is not an element of y then, y must be an element of y. In other words, if we assume the set y exists we get a contradiction. Therefore we have to conclude that y does not exist. Using similar reasoning one can also show that the set of all sets does not exist either (see proof later in this document). But this raises deep questions:

1. What does it mean to say that a set exists or does not exist? For example Leopold Kronecker, a German mathematician born in 1823, claimed that the only numbers that assuredly exist are the natural numbers (1,2,3 ...). According to him the set of real numbers are just a fantasy that does not exist. But think about it, what criteria, other than authority, can we use to decide whether the natural numbers or the real numbers exist?
2. How can we tell whether something is or is not a set?
3. What are valid elements of a set?

Axiomatic set theory was developed to provide answers to such questions. In axiomatic set theory:

1. A set exists if the proposition that asserts its existence is logically true. Moreover within this theory there are only sets so if a formal object is not a set, it does not exist.
2. If the assumption that an object exists leads to a contradiction we can assert that that object does not exist, or equivalently, that it is not a set.
3. There are no atomic elements: An object exists if and only if it is a set. Of course sets can have elements but those elements must be sets themselves otherwise they would not exist.

One "annoying" aspect of axiomatic set theory is that sets become a logical abstraction detached from our everyday experience with collections of physical objects. You should think of mathematical sets as logical "objects" which are part of a formal structure. Within this theory to say that an object exists is the same as saying that it is a set. To say that an object does not exist is the same as saying that it is not a set. Everyday collection of physical objects are no longer sets in this theory and thus they do not exist. While this approach may strike you as peculiar, it turns out to be extremely powerful and in fact it has become the foundation of mathematics. The formal structure of set theory while independent from the physical world provides very useful tools to model the world itself. The key is to develop set structures constrained in ways that mirror essential properties of the physical world. For example, the properties of the set of natural numbers (i.e., 1,2,3, $\cdots$) mirrors our experience counting collections of physical objects.

Axiomatic set theory is a first order logical structure. First order logic works with propositions, i.e., logical statements constructed according to the rules of logic and that can take two values. For convenience we call these two values "True" and "False". Set theory, and thus the entire body of mathematics reduces to logical propositions that use the following elements:

1. Variables (e.g., $a, b, \cdots x, y, z$) which stand for sets.
2. The predicate $\in$, which stands for element inclusion. For example, if the proposition $(x \in y)$ takes the value true, we know that both x and y are sets and that x is an element of y. For example, the proposition

$$\{1, 2, 3\} \in \{\{1, 2\}, \{4, 5\}, \{1, 2, 3\}\} \tag{3}$$

   takes the value "True".
3. Logical operators
   (a) $\neg P$, where $\neg$ is the logical "negation" operator.
   (b) $P \wedge P$, where $\wedge$ is the logical "and" operator.
   (c) $P \vee P$, where $\vee$ is the logical "or" operator.
   (d) $P \rightarrow P$, where $\rightarrow$ is the logical "implication" operator.
   (e) $P \leftrightarrow P$, where $\leftrightarrow$ is the logical "bijection" operator.
   (f) $\forall x P$ is the logical "for-all" quantifier.
   (g) $\exists x P$ is the logical "exists" quantifier.

The names of the different operators (i.e., "negation", "and", "or", "implication" ...) are selected for convenience. We could have given them completely different names, all we really need to know is how they operate on propositions.

All propositions in set theory are built out of atomic propositions of the form $(x \in y)$ connected using the logical operators. If P and Q are propositions, e.g., P could be $(x \in y)$ and Q could be $(y \in z)$ then $\neg P$, $P \wedge P$, $P \vee Q$ $P \rightarrow Q$, $P \leftrightarrow Q$, $\forall x P$ and $\exists x P$ are also propositions.

The effect of the connectives on the truth value of propositions is expressed in Table .

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Table 1: The truth tables of the logical operators. T stands for "True" and F for "False".

Thus, if the proposition P takes value "True" and the proposition Q takes the value "False" then the proposition $(P \wedge Q)$ takes the value "False". The propositions $\forall x P$ and $\exists x P$ tell us that x is a variable that can take as value any formal object that qualifies as a set. It also tells us that P is a proposition whose truth value depends on x. For example,

P could be $(x \in y) \vee (x \in z)$, where y and z are fixed sets and x acts as a variable. The proposition $\forall x P$ takes the value "True" if P takes the value "True" for all sets. The proposition $\exists x P$ takes the value "True" if there is at least one set for which P takes the value "True". Remember when we say for all sets we do not mean sets of physical objects. In fact we still have to define what we mean by set.

0.1 Proofs and Logical Truth

Sometimes we treat propositions as formulas whose truth value depends on the truth values taken by variables in the proposition. For example if P and Q are propositional variables then the $P \wedge Q$ is a propositional formula whose truth value depends on the specific truth values taken by P and Q. We say that a propositional formula is **logically true** if for all valid combinations of truth values of the elements in the formula, the formula can only take the value "True". For example for the formula $(P \vee \neg P)$ there is only two valid combination of truth values for P and $\neg P$: "True, False" and "False, True". In both case the formula $(P \vee (\neg P))$ takes the value "True" and thus we say that it is logically true. Similarly if a propositional formula can only take "False" values we say that it is **logically false**. For example $(P \wedge (\neg P))$ is logically false. A **proof** is a process that shows a propositional formula is logically true.

1 The Axioms of Set Theory

To simplify the presentation of axiomatic set theory I will use "pseudocode", i.e., a combination of logical propositions, mathematical symbols, and English statements. I do so under the understanding that all these statements can be written as pure logical propositions.

I will use the symbol $\notin$ in propositions of the form $(x \notin y)$ as an alternative notation to $\neg(x \in y)$. I will use the formula

$$\exists\{x : P\} \tag{4}$$

as an alternative notation to the propositional formula

$$\exists y \forall x P \tag{5}$$

This formula simply says that there is a set of elements that satisfy the proposition P. If the formula takes the value "True" then the symbols $\{x : P\}$ refers to a set that make the proposition $\forall x P$ "True". When a set x makes the proposition P true, I will say that x satisfies P. For example the set 1 satisfies the propositional formula $(x \in \{1, 2\})$.

In set theory all existing objects are sets. If an object exists it is a set otherwise it does not exist. To remind us of the fact that sets include elements we sometimes refer to sets as a collection of sets, or as a families of sets. This is just a "human factors" trick since the theory makes no distinction between sets, families, collections or elements. In axiomatic set theory elements, collections, and families are just sets.

Axiomatic set theory is commonly presented using 9 redundant axioms, which are the foundation of all mathematical statements.

1.1 Axiom of Existence:

An axiom is a restriction in the truth value of a proposition. The axiom of existence forces the proposition

$$\exists y \forall x (x \notin y) \tag{6}$$

to take the value "True". We call call the sets that satisfy $\forall x(x \notin y)$ **empty sets**. Thus the axiom of existence tells us that there is at least one empty set, we will see later that in fact there is only one empty set.

1.2 Axiom of Equality:

This axiom is also called the axiom of extensionality and it defines the predicate "=". For mnemonic convenience when the proposition $(x = y)$ takes the value "True" we say that the sets x and y are equal. In order to define how the symbol "=" works it is convenient to create a new predicate, which we will symbolize as $\subset$. The new predicate works as follows: For all sets u and v if the proposition

$$\forall x(x \in u) \rightarrow (x \in v) \tag{7}$$

is true then the proposition $(u \subset v)$ is true. For mnemonic convenience if the proposition $(u \subset v)$ takes the value "True" we say that u is a subset of v.

The axiom of equality says that if the proposition $(u \subset v) \wedge (v \subset u)$ is true then the proposition $(u = v)$ is true. In other word, the proposition

$$\forall u(u \in x \leftrightarrow u \in y) \rightarrow (x = y) \tag{8}$$

takes the value "True". The formula $(x \neq y)$ is used as an alternative notation to $\neg(x = y)$. We will now use the axiom of equality to prove that there is only one empty set.

Theorem: The empty set is unique.

Proof: Let x and y be empty sets, then $u \in y$ and $u \in x$ are always false for all sets u. Thus $(u \in y \leftrightarrow u \in x)$ is true for all sets u and since by the axiom of equality

$$(\forall u(u \in x \leftrightarrow u \in y)) \rightarrow (x = y) \tag{9}$$

is true then it follows that $(x = y)$ must be true. Hereafter we identify the empty set with the symbols $\emptyset$ or alternatively with the symbol $\{\}$.

$\square$

1.3 Axiom of Pair:

So far set theory has only given us one set: the empty set. The axiom of pair brings new sets to life. The axiom of pair says that if x and y exist (i.e., if they are sets) there also exists a set whose only elements are x and y. We will represent such set as $\{x, y\}$. The axiom of pair forces the proposition

$$\forall x \forall y \exists \{x, y\} \tag{10}$$

to take the value "True". The set made out of the sets a and a is symbolized as $\{a, a\}$ or $\{a\}$ and is called the singleton whose only element is a. So starting with the empty set $\emptyset$, it follows that the set $\{\emptyset\}$ exists. Note that $\emptyset$ and $\{\emptyset\}$ are different since the first has no element and the second has one element, which is the empty set.

Ordered pairs: The ordered pair of the sets x and y is symbolized (x, y) and defined as follows

$$(x, y) \triangleq \{\{x\}, \{x, y\}\} \tag{11}$$

where $\triangleq$ stands for "equal by definition".

Exercise: Prove that two ordered pairs (a, b) and (c, d) are equal if and only if $a = b$ and $c = d$.

Ordered sequences: Let $x_1 \cdots x_n$ be sets. The ordered sequence $(x_1, \cdots, x_n)$ is recursively defined as follows

$$(x_1, \cdots, x_n) = ((x_1, \cdots, x_{n-1}), x_n) \tag{12}$$

Exercise: Prove that two n-tuples pairs $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are equal if and only if $a_1 = b_1$ and $a_2 = b_2$ and ... $a_n = b_n$.

1.4 Axiom of Separation:

This axiom tells us how to generate new sets out of elements of an existing set. To do so we just choose elements of an existing set that satisfy a proposition. Consider a proposition P whose truth value depends on the sets u and v, for example, P could be $(u \in v)$. The axiom of separation forces the proposition

$$\exists \{x : (x \in u) \wedge P\} \tag{13}$$

to take the value "True" for all sets u, v and for all propositions P with truth values dependent on u and v.

Fact: There is no set of all sets.
The proof works by contradiction. Assume there is a set of all sets, and call it u. Then by the axiom of separation the following set r must exist

$$r = \{x : (x \in u) \wedge (x \notin x)\} \tag{14}$$

and since $(x \in u)$ is always true, this set equals the set

$$\{x : x \notin x\} \tag{15}$$

Then $(r \in) r \leftrightarrow (r \notin r)$ which is a logically false proposition. Thus the set of all sets does not exist (i.e., it is not a set).

Intersections: The intersection of all the sets in the set s, or simply the intersection of s is symbolized as $\cap s$ and defined as follows:

$$\cap s = \{x : \forall y(\ (y \in s) \rightarrow (x \in y) \ )\} \tag{16}$$

For example, if $s = \{\{1, 2, 3\}, \{2, 3, 4\}\}$ then $\cap s = \{2, 3\}$. The axiom of separation tells us that if s exists then $\cap s$ also exists. We can then use the axiom of equality to prove that $\cap s$ is in fact unique. For any two sets x and y, we represent their intersection as $x \cap y$ and define it as follows

$$x \cap y \triangleq \cap \{x, y\} \tag{17}$$

For example, if $x = \{1, 2, 3\}$ and $y = \{2, 3, 4\}$ then

$$x \cap y = \cap \{\{1, 2, 3\}, \{2, 3, 4\}\} = \{2, 3\} \tag{18}$$

1.5 Axiom of Union:

It tells us that for any set x we can make a new set whose elements belong to at least one of the elements of x. We call this new set the union of x and we represent it as $\cup x$. For example, if $x = \{\{1, 2\}, \{2, 3, 4\}\}$ then $\cup x = \{1, 2, 3, 4\}$. More formally, the axiom of union forces the proposition

$$\forall s \exists \cup s \tag{19}$$

to be true. Here $\cup s$ is defined as follows

$$\cup s \triangleq \{x : \exists y (y \in s) \wedge (x \in y)\} \tag{20}$$

For example, if $x = \{\{1, 2, 3\}, \{2, 3, 4\}\}$ then $\cup x = \{1, 2, 3, 4\}$. Using the axiom of equality $\cup x$ can be shown to be unique. For any two sets x and y, we define the union of the two sets as follows For example,

$$\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\} \tag{21}$$

$$x \cup y \triangleq \cup \{x, y\} \tag{22}$$

1.6 Axiom of Power:

This axiom tells us that for any set x the set of all subsets of x exists. We call this set the power set of x and represent it as $\mathfrak{P}(x)$. More formally, the axiom of power forces the proposition

$$\forall s \exists \{x : x \subset u\} \tag{23}$$

to take the value "True". For example, if $s = \{1, 2\}$ then

$$\mathfrak{P}(s) = \{\{1\}, \{2\}, \emptyset, \{1, 2\}\}. \tag{24}$$

Cartesian Products: The Cartesian product of two sets u and v, is symbolized $a \times b$ and defined as follows

$$a \times b = \{(x, y) : (x \in a) \wedge (y \in b)\} \tag{25}$$

Using the axioms of separation, union and power, we can show that $x \in y$ exists because it is a subset of $\mathfrak{P}(\mathfrak{P}(x \cup y))$. Using the axiom of identity we can show that it is unique.

Functions: A function f with domain u and target v is a subset of $u \times v$ with the following property: If (a, c) and (b, c) are elements of f then $a = b$. More formally, if the proposition

$$\forall a \forall b \forall c (((a, c) \in f) \wedge ((b, c) \in f) \rightarrow (a = b)) \tag{26}$$

takes the value "True" then we say that the set f is a function.

The following formulae are alternative notations for the same proposition:

$$(x, y) \in f \tag{27}$$

$$y = f(x) \tag{28}$$

$$x \mapsto f(x) \tag{29}$$

$$x \mapsto y \tag{30}$$

1.7 Axiom of Infinity:

This axiom forces the proposition

$$\exists s \forall x (x \in s) \rightarrow (\{x, \{x\}\} \in s) \tag{31}$$

to take the value "True". In English this axiom tells us that there is a set s such that if x is an element of s then the pair $\{x, \{x\}\}$ is also an element of s. Sets that satisfy the proposition

$$\forall x (x \in s) \rightarrow (\{x, \{x\}\} \in s) \tag{32}$$

are called inductive (or infinite) sets.

Natural numbers: The natural numbers plus the number zero are defined as the intersection of all the inductive sets and are constructed as follows:

$$0 \triangleq \{\} \tag{33}$$

$$1 \triangleq \{0, \{0\}\} = \{\{\}, \{\{\}\}\} \tag{34}$$

$$2 \triangleq \{1, \{1\}\} = \{\{\{\}, \{\{\}\}\}, \{\{\{\}, \{\{\}\}\}\}\} \qquad \cdots \tag{35}$$

The axiom of existence in combination with the axiom of infinity guarantee that these sets exist. Note that the symbols $1, 2, \cdots$ are just a mnemonic convenience. The bottom line is that numbers, and in facts all sets, are just a bunch of empty curly brackets!

1.8 Axiom of Image:

Let $f : u \to v$ be a function (i.e, a subset of $u \times b$). Define the image of u under f as the set of elements for which there is an element of u which projects into that element. We represent that set as $I_f(u)$. More formally

$$I_f(u) = \{y : \exists x(x \in u) \wedge (f(x) = y)\} \tag{36}$$

The axiom of image, also called the axiom of replacement, tells us that for all sets u and for all functions f with domain u the set $I_f(u)$ exists.

1.9 Axiom of Foundation:

This axiom prevents the existence of sets who are elements of themselves.

1.10 Axiom of Choice:

This axiom tells us that every set with no empty elements has a choice function. A choice function for a set s is a function with domain s and such that for each $x \in s$ the function takes a value $f(x) \in x$ which is an element of x. In other words, the function f picks one element from each of the sets in s, thus the name "choice function". For example, For the set $s = \{\{1, 2, 3\}, \{2, 5\}, \{2, 3\}\}$ the function $f : s \to \{1, 2, 3\}$ such that

$$f(\{1, 2, 3\}) = 3 \tag{37}$$

$$f(\{2, 5\}) = 2 \tag{38}$$

$$f(\{2, 3\}) = 2 \tag{39}$$

is a choice function since for each set in s the function f picks an element of that set. The axiom of choice is independent of the other axioms, i.e., it cannot be proven right or wrong based on the other axioms. The axiomatic system presented here is commonly symbolized as ZFC (Zermelo-Fraenkel plus axiom of Choice), the axiomatic system without the axiom of choice is commonly symbolized as ZF.

2 History

- The first version of this document was written by Javier R. Movellan in 1995. The document was 8 pages long.
- The document was made open source under the GNU Free Documentation License Version 1.1 on August 9 2002, as part of the Kolmogorov project.
- October 9, 2003. Javier R. Movellan changed the license to GFDL 1.2 and included an endorsement section.