# Calculus of Variations

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Our goal is to find a function $f$ that minimizes the following function

$$I(f) = \int_a^b F_t(f(t), f'(t))dt \tag{1}$$

where

$$f'(x) \overset{\text{def}}{=} \frac{df(x)}{dx} \tag{2}$$

We will find a necessary condition for $f$ to be at a minimum. The condition is equivalent to the zero gradient condition for the discrete time case.

Let $f\hat{f}$ be a local minimum of $F$. Let now define a family of functions of the following form

$$h(t) = \hat{f}(t) + \epsilon\eta(t) \tag{3}$$

where $\eta$ is an arbitrary function with continuous second partial derivatives and such that $\eta(a) = \eta(b) = 0$. Let

$$\tilde{I}(\epsilon) = I((h(\cdot, \epsilon)) = \int_a^b F_t(h(t), h'(t))dt \tag{4}$$

For $\hat{f}$ to be a minimum point we require

$$\left.\frac{d\tilde{I}(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0 \tag{5}$$

thus

$$\left.\int_a^b \left( \frac{dF_t(h(t), h'(t))}{dh(t)}\eta(t) + \frac{dF_t(h(t), h'(t))}{dh'(t)} \frac{dh'(t)}{d\epsilon} \right) dt \right|_{\epsilon=0} = 0 \tag{6}$$

Note for a fixed $\hat{f}$

$$\frac{dh'(t)}{d\epsilon} = \epsilon\eta'(t) \tag{7}$$

for $\epsilon = 0$

$$\frac{dF_t(h(t), h'(t)}{dh(t)} = \frac{dF_t(\hat{f}(t), \hat{f}'(t)}{d\hat{f}(t)} \tag{8}$$

$$\frac{dF_t(h(t), h'(t)}{dh'(t)} = \frac{dF_t(\hat{f}(t), \hat{f}'(t)}{d\hat{f}'(t)} \tag{9}$$

Moreover, using integration by parts

$$\int_a^b \frac{dF_t(\hat{f}(t), \hat{f}'(t))}{d\hat{f}'(t)}\eta'(t)dt = \left. F_t(\hat{f}(t), \hat{f}'(t))\eta(t) \right|_a^b \tag{10}$$

$$- \int_a^b \frac{d}{dt} \left( \frac{dF_t(\hat{f}(t), \hat{f}'(t)}{d\hat{f}'(t)} \right) \eta(t)dt \tag{11}$$

$$= - \int_a^b \frac{d}{dt} \left( \frac{dF_t(\hat{f}(t), \hat{f}'(t)}{d\hat{f}'(t)} \right) \eta(t)dt \tag{12}$$

because $\eta(a) = \eta(b) = 0$. Thus

$$\left.\int_a^b \left( \frac{dF_t(h(t), h'(t))}{dh(t)} - \frac{d}{dt} \left( \frac{dF_t(\hat{f}(t), \hat{f}'(t))}{d\hat{f}'(t)} \right) \right) \eta(t)dt \right|_{\epsilon=0} = 0 \tag{13}$$

The fundamental lemma of calculus of variations states that if

$$\int_a^b m(x)g(x)dx = 0 \tag{14}$$

for all $g$ with continuous second partial derivatives, then

$$m(x) = 0, \text{ for } x \in (a, b) \tag{15}$$

Applying the Lemma to our case, we get that

$$\frac{dF_t(h(t), h'(t))}{dh(t)} - \frac{d}{dt}\left(\frac{dF_t(\hat{f}(t), \hat{f}'(t))}{d\hat{f}'(t)}\right) = 0, \quad \text{for } t \in (a, b) \tag{16}$$

This is called the Euler-Lagrange differential equation.