

Pneumatic Cylinder Model

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A pneumatic cylinder is a device with two chambers separated by a bore. Each chamber has 1 valve and each valve has two ports (orifices): One port (the supply port) connects the chamber to a compressor and the other port (the exhaust port) connects the chamber to atmospheric pressure. A small cylinder typically has a bore diameter of 5/16 and a stroke of 2 inches. Larger, yet typical cylinders have bore diameters of 3 inches stroke lengths of about 6 inches. They typically rate for pressures of up to 250 Pounds per Square Inch (psi). A cylinder with a 3 inch diameter bore running at 100 psi produces a force of $100 \pi (1.5)^2 = 707$ force pounds . A cylinder with a 5/16 bore running at 100 psi produces 7.7 force pounds. These are very large numbers considering the simplicity of the system and the lack of friction causing gears. The large force to friction ratio is one of the main advantages of pneumatic actuators when compared to electric motors. The disadvantage is the fact that pneumatic actuators are much harder to control using classical controllers than electric motors.

Here we present a simple yet quite useful cylinder model. The model is intended for experimentation with modern control methods applied to pneumatic control. The model consists of two chambers assumed to have uniform pressure across the entire volume, and 2 constant pressure sources (the compressor and the atmosphere). The key part of the cylinder model is the port, i.e, the orifice connecting two volumes that have different pressures. The port model describes the flow of fluid (air) through the port as a function of the area of the orifice, and the upstream and downstream pressures.

1 Port Model

In this section we present the model of the port and in later sections we apply the port model to the overall cylinder model. A port is just an orifice connecting two chambers with different pressures (see Figure 1). Key assumptions in this model are that the area of the port is small, that the plate separating the chambers is thin, that the fluid is a perfect gas, that the temperatures in the two chambers are equal, and that there is no temperature transfer (heat production) in the process.

The port model describes the movement of fluid that occurs when connecting two chambers (upstream and downstream chambers) of different pressure via a small orifice. Given the assumptions described above it can be shown that the flow \dot{m} of fluid mass through the orifice is described by the

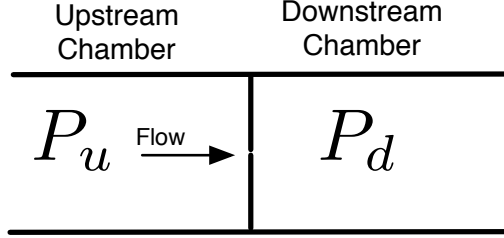


Figure 1: *Thin Plate Port Model*

following equation:

$$\dot{m} = \begin{cases} a f(P_u, P_d) & \text{if } P_u \geq P_d \\ -a f(P_d, P_u) & \text{if } P_u < P_d \end{cases} \quad (1)$$

where a is the area of the orifice and

$$f(P_u, P_d) = \begin{cases} \alpha P_u \sqrt{\left(\frac{P_d}{P_u}\right)^{\frac{2}{k}} - \left(\frac{P_d}{P_u}\right)^{\frac{k+1}{k}}} & \text{for } P_u/P_d \leq \theta \\ \beta P_u & \text{for } P_u/P_d > \theta \end{cases} \quad (2)$$

where

$$\alpha = c \sqrt{\frac{2M}{ZRT} \frac{k}{k-1}} \quad (3)$$

$$\beta = c \sqrt{\frac{kM}{ZRT} \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \quad (4)$$

$$\theta = \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \quad (5)$$

and the M, Z, R, T, k, c constants are defined in Table 1.

Figure 2 shows the air flow as a function of the pressure in one of the chambers (the other chamber is at atmospheric pressure) and the diameter of the orifice. The function is fact continuous, and its first derivative is also continuous. When the upstream pressure is larger than a constant θ times the downstream pressure, the flow is independent of the downstream

Mass Flow	\dot{m}	Kg/s
Area of hole	A	m^2
Gas Molecular Mass	M	0.029 for air, Kg/mol
Temperature	T	K°
Upstream Pressure	P_u	$Pascals$
Downstream Pressure	P_d	$Pascals$
Universal Gas Law Constant	R	8.314472 ($Pascal\ m^3$)/($mol\ K^\circ$)
Discharge coefficient	c	0.72, dimensionless
Gas compressibility Factor	Z	0.99 for air, dimensionless
Specific Heat Ratio	k	1.4 for air, dimensionless

Table 1: Parameters of thin plate port model.

pressure. When this occurs, we say the flow is *choked*, or equivalently, that the flow has sonic velocity. Choked flow is caused by the Venturi effect. When a flowing fluid at a certain pressure and temperature flows through a restriction (such as the hole in an orifice plate or a valve in a pipe) into a lower pressure environment, under the conservation of mass the fluid velocity must increase for initially subsonic upstream conditions as it flows through the smaller cross-sectional area of the restriction. At the same time, the Venturi effect causes the pressure to decrease. Choked flow is a limiting condition which occurs when the velocity will not increase with a further decrease in the downstream pressure environment. The physical point at which the choking occurs is when the exit plane velocity is at sonic conditions or at a Mach number of 1. Note that while in choked conditions the flow rate is independent of the downstream pressure, it is still dependent on the upstream pressure. Choked flow is in many cases a desirable property for it decouples the flow from the downstream conditions facilitating the design of pneumatic circuits.

2 Two Chamber Cylinder

In the case of the 2-chamber cylinder we have one valve per chamber and each valve has 2 ports, a supply port that connects to the compressor and an exhaust port that connects to the atmosphere. Let a_{ij} represent the area of port j from chamber i . For each chamber i we let a_{i1} represent the area

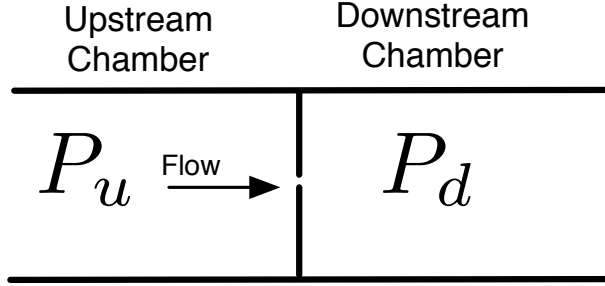


Figure 2: *The Thin Plate Flow Function.* We measure air flow (in cubic feet per minutes) as a function of pressure in Chamber A. Chamber B is at atmospheric pressure (14.5 PSI). The figure shows the flow rate for 4 different orifice diameters (in millimeters). The model parameters were as follows: $M = 0.029 \text{ Kg/mol}$, $R = 8.314 \text{ Pa m}^3/(\text{mol Kelvin})$, $c = 0.72$, $k = 1.4$, $Z = 0.9987$. To convert from air mass flow into air volume flow we used an air density of $\rho = 1.2 \text{ Kg/m}^3$. Note were Chamber B is at less of atmospheric pressure (vacuum) the flow is in the negative direction. The 4 different curves display different values of the diameter of the orifice.

of the supply port and a_{i2} the exhaust port (see Figure 3). The flow of air through each port is governed by the thin port equation

$$\dot{m}_{ij} = a_{ij} f(Q_j, P_i) \quad (6)$$

where f is the thin port flow function (2), P_i is the pressure of chamber i , Q_1 is the supply pressure typically provided by a compressor running at 90 PSI. Q_2 is the exhaust pressure, typically atmospheric pressure at about 14.5 PSI. Note we let Q_j be the upstream pressure and P_i the downstream pressure, thus the flow is positive if it moves into a cylinder's chamber i and negative if it moves out of a cylinder's chamber. The total flow of air mass into chamber i is simply the sum of the flow through its two ports:

$$\dot{m}_i = \sum_{j=1}^2 \dot{m}_{ij} \quad (7)$$

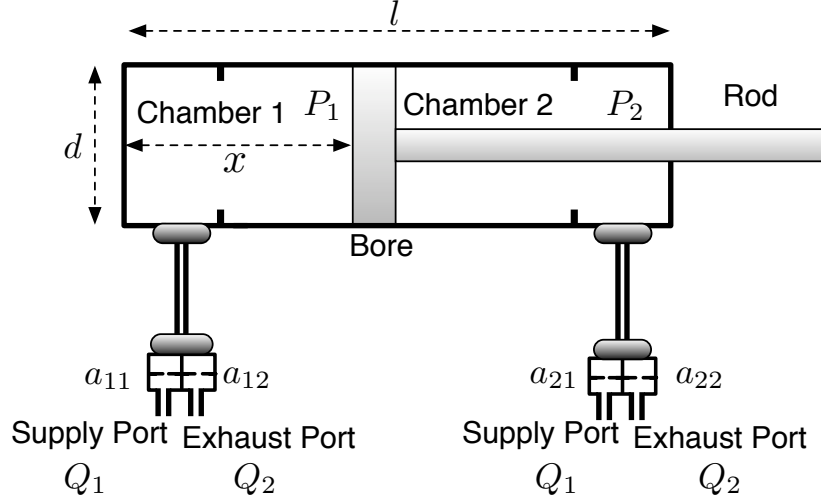


Figure 3: *Cylinder Model*

2.1 Chamber Pressure Dynamics

As air moves into a chamber the pressure of the chamber increases. This moves the bore. The pressure increases for the chamber the bore moves into and decreases for the chamber it moves away from. Using the law of conservation of energy, assuming a constant temperature T and no heat exchange, we get that the velocity of the chamber pressure is as follows

$$\dot{P}_1 = k \frac{R T \dot{m}_1 - P_1 A \dot{x}}{V_1} \quad (8)$$

$$\dot{P}_2 = k \frac{R T \dot{m}_2 + P_2 A \dot{x}}{V_2} \quad (9)$$

where \dot{x} is the velocity of the bore, and V_i is the volume of chamber i

$$V_1 = x A \quad (10)$$

$$V_2 = (l - x) A \quad (11)$$

$$A = \pi \left(\frac{d}{2} \right)^2 \quad (12)$$

a_{ij}	Area of port j in chamber i . Let $j = 1$ supply, $j = 2$ exhaust	m^2
\dot{m}_{ij}	Air mass flow into chamber i via port j	Kg/s
V_i	Volume of chamber i	m^3
l	cylinder length	m
d	Bore Area	m^2
x	position of Bore	m
P_i	Pressure in chamber i	Pascal
Q_1	Exhaust Pressure (atmosphere)	Pascal
Q_2	Supply Pressure	Pascal
k_s	Static Friction	Newtons
k_d	Dynamic Friction	Newtons
k_v	Viscosity Coefficient	<i>Newtons/(m/s)</i>

Table 2: Symbols used for cylinder model

2.2 Force Dynamics

We have four types of forces affecting x . First there is the force due to the pressure in the two chambers.

$$F_1 = (P_1 - P_2) A \quad (13)$$

Second we have the dynamic friction force and viscous forces

$$F_2 = \text{sgn}(\dot{x})k_d + k_v\dot{x} \quad (14)$$

Third we have the load, F_3 , i.e., the external force applied to the piston. Finally we need to consider the static friction force k_s . The static friction force counteracts all the other forces up to a threshold k_s

$$F_4 = \begin{cases} -(F_1 + F_2 + F_3), & \text{if } |F_1 + F_2 + F_3| < k_s \\ 0, & \text{else} \end{cases} \quad (15)$$

We can then apply Newton's law to complete the model dynamics:

$$\ddot{x} = \frac{1}{m} (F_1 + F_2 + F_3 + F_4) \quad (16)$$

2.3 Computer Simulations

Model Parameters The model parameters are described in Table 2. The properties of the air, the temperature of the system, the pressure of the compressor, the atmospheric pressure, the size of the cylinder, and the friction coefficients determine the different model parameters.

State Variables The state variables include the chamber pressures P_1, P_2 , the location of the bore x , and the bore velocity \dot{x} . The control variables are the apertures of the four ports: $a_{11}, a_{1,2}, a_{21}, a_{22}$. These are typically controlled by an electric signal and the mechanism transforming voltages into port apertures typically has a bandwidth of 100 Hz.

State Dynamics To simplify the presentation we will use a frictionless model moving a load of constant mass m . In digital computers we simulate the process in discrete time using small time steps δ . First we update the state of the bore and rod location and velocity

$$\ddot{x}_t = \frac{A}{m}(P_1(t) - P_2(t)) \quad (17)$$

$$\dot{x}_{t+\delta} = \dot{x}_t + \ddot{x}_t \delta \quad (18)$$

$$x_{t+\delta} = x_t + \dot{x}_t \delta + \frac{1}{2} \ddot{x}_t \delta^2 \quad (19)$$

where $A = \pi \left(\frac{d}{2}\right)^2$. Second we update the chamber pressure variables

$$\dot{m}_1(t) = a_{11}(t)f(Q_1, P_1(t)) + a_{12}(t)f(Q_2, P_1(t)) \quad (20)$$

$$\dot{m}_2(t) = a_{21}(t)f(Q_1, P_2(t)) + a_{22}(t)f(Q_2, P_2(t)) \quad (21)$$

$$\dot{P}_1(t) = k \frac{RT\dot{m}_1 - P_1(t)A\dot{x}_t}{x_t A} \quad (22)$$

$$\dot{P}_2(t) = k \frac{RT\dot{m}_2 + P_2(t)A\dot{x}_t}{(l - x_t)A} \quad (23)$$

$$P_1(t + \delta) = P_1(t) + \dot{P}_1(t)\delta \quad (24)$$

$$P_2(t + \delta) = P_2(t) + \dot{P}_2(t)\delta \quad (25)$$

Typical Parameter Values:

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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Cylinder Parameters %%%%%%%%%%%%%%%
sP = 100; % supply (compressor) pressure (Psi)
eP = 14.5; % exhaust (atmospheric) pressure (Psi)
T = 70; % Temperature in Farenheit

kS= 0; %100; % Static friction of piston in Newtons
kV = 100;% 100; % Viscosity friction coefficient of piston in N/(m/s)
kC = kS/2; % Coulomb friction of piston in Newtons

Br= 1; % Bore radius (inches)
l = 2; % Stroke length (inches)

m = 4; % Mass of bore + load in Kg

x = 0; % location of bore from origin of chamber 1, (in inches)

bl = 1.25; % length of bore in inches ( irrelevant if l is known)
cl = l+ bl; %cylinder length (irrelevant if l is known)

rV(1) = 1*pi*(0.6^2); % Reservoir for chamber 1 (cubic inches)
rV(2) = 0.75*pi*(0.35^2); % reservoir for chamber 2 (cubic inches)

% The reservoir is the minimum volume of the chamber when the bore is
% pushed all the way against the chamber. Thus a chamber can never have
% zero volume

dt= 0.00001; % time step in seconds
          % iterate through time

graphDt= 2/300; % time in simulated seconds between graph updates

initGraphs;

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%%%%%%%%%% Gas Parameters and Conversions %%%%%%%%%%
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sP = sP*6894.76; %Convert Psi to Pascals
eP = eP*6894.76; %Convert Psi to Pascals
rMax = rMax*0.0254; % Convert inches to meters
Br = Br*0.0254; % Convert inches to meters
l = l *0.0254; % Convert inches to meters
x = x *0.0254; % Convert inches to meters
bl = bl*0.0254; % Convert inches to meters
cl = cl*0.0254; % Convert inches to meters
rV(1) = rV(1)*(0.0254^3); % from cubic inches to cubic meters
rV(2) = rV(2)*(0.0254^3); % from cubic inches to cubic meters
BoreArea = pi*Br^2; % in square meters
T= 5/9 * (T - 32) + 273 ;% Convert Farenheit to Kelvin
M = 0.029; % air molecular mass (Kg/mol)
rho = 1.2; % density of air (Kg/cubic meter)
R = 8.314472; % Universal gas law constant in (Pa m3)/ (mol Kelvin)
c= 0.72 ; % Discharge coefficient (dimensionless)
k = 1.4 ;% specific heat ratio of air (dimensionless)
Z = 0.9987; % air compressibility factor at p= 5 bars and T=300K, dimensionless
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alpha = c *sqrt(2*M*k/(Z*R*T*(k-1)));
beta = c *sqrt( k*M/(Z*R*T) *((2/(k+1))^(k+1)/(k-1)));
theta = ((k+1)/2)^(k/(k-1));
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3 Compliance

Stiffness, is the resistance of a body to deformation. Compliance is the inverse of stiffness. Consider a 2 chamber cylinder at equilibrium with all its ports closed. Thus the two chambers have equal pressure P

$$P_1 = P_2 = P \tag{26}$$

We can measure the stiffness of a cylinder by calculating the force excised by the piston when we push it by a small amount δ . Let P'_1, P'_2, V'_1, V'_2 be the

chamber pressures and chamber volumes after we pushed the cylinder. Note

$$\frac{P'_1}{P_1} = \frac{V_1}{V'_1} = \frac{x}{x - \delta} \quad (27)$$

$$\frac{P'_2}{P_2} = \frac{V_2}{V'_2} = \frac{l - x}{l - x + \delta} \quad (28)$$

$$(29)$$

Thus the force F produced by the piston in response to the push is

$$F = A(P'_1 - P'_2) = AP \left(\frac{x}{x - \delta} - \frac{l - x}{l - x + \delta} \right) \quad (30)$$

Note this force, the stiffness, increases with the initial pressure P in the two chambers. Thus it is possible to use pressure to control the stiffness (compliance) of the cylinder.

4 Appendix

4.1 Units of Force

- A **Newton** is the force needed to accelerate one Kg of Mass 1 *meter/sec*². On Earth 1 Kg of Mass exerts a force of 9.81 Newtons. On Earth you can think of a Newton as approximately 100 grams (3.5 ounces, 0.22 pounds).
- A **Pound Force** is the Earth's gravitational force on 1 pound of mass.
- A **Kilopond** is the Earth's gravitational force on a Kilogram of mass. One Kilopond is 9.81 Newtons or 2.2046 Pounds of force.

4.2 Units of Pressure

- **Pascal (Pa)**: One Newton per square meter. (SI standard).
- **Mega Pascal** One Million pascals. 100 psi = 0.6894 Mega Pascals
- **Barye (Ba)**: One dyne per square centimeter (cgs standard)
- **PSI** A pound of force per square inch. 1 Psi = 6894.76 Pa

- **PSIG and PSIA** Pound force per square inch gauge. A unit of pressure relative to the surrounding atmosphere. By contrast, psia (pound-force per square inch absolute) measures pressure relative to a vacuum (such as that in space). At sea level, Earth's atmosphere actually exerts a pressure of 14.7 psi (see below). If a pressure gauge is calibrated to read zero in space, then at sea level on Earth it would read 14.7 psi. Thus a reading of 30 psig, on Earth, on a tire gauge represents an absolute pressure of 44.7 psi. Bike tires are at about 60 psi, about 4 bars. Typical commercial compressors run at at about 100 PSI. High pressure compressors go up to 6000 psi (415 bars). Pneumatic cylinders run at up to 250 Psi
- **Bar** a SI recognized unit popular in Europe. 1 bar = 10^5 Pascals = 10 Nsc= = 14.50377 Psi.

4.3 Units of Flow

- **Cubic Feet per Minute** (CFM or CFPM) Typical airflow requirement of a pneumatic hand tool is 10 CFM (cubic feet per minute) at 90 PSI. 1 Cubit Foot = 28.3 Liters. 10 Cubic Foot per Minute = 4.7 liters per second. Pneumatic drills, consume about 4 CFM. Standard compressors deliver from 4 to 12 CFM at 90 PSI.
- **Standard Cubic Feet per Minute.** A CFPM on dry air at 70 degrees Farenheit and 14.7 PSIA.
- **Litters per Second:** 4.7 liters per second make 1 cubic foot per minute.

4.4 Pneumatic Valves

Figure 4 shows a standard implementation of a pneumatic valve. As a spool move left and right, it opens and closes the exhaust and supply ports to a cylinder's chamber.

Figure 5 shows a standard schematic representation of a 5/3 valve (5 ports 3 end states). The schematic has 3 boxes representing 3 states. The arrows represent the direction of flow. Pointing from high to low pressure ports. The T represent a closed port. The triangles are for exhaust ports.

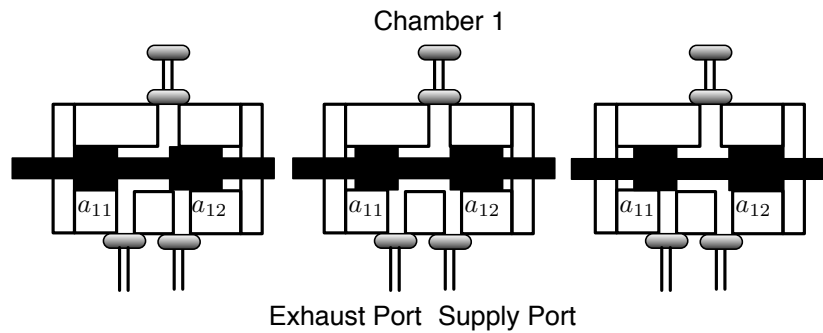


Figure 4: *Schematics of a proportional valve. Left: Exhaust Port Open Supply Port Closed.Center: Exhaust and Supply Partially Open. Right: Exhaust Closed Supply Open.*

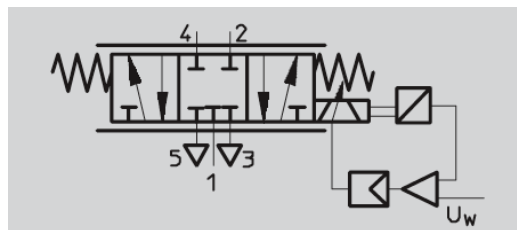


Figure 5: *Schematics of a 5/3 pneumatic valve (5 ports, 3 states)*

The 3 boxes represent the three main states the valve may be on. Ports 5,2, In State 1 (left-side box) Port 5 is closed, port 4 connects to the supply and port 2 connects to the exhaust. In State 2 (center) all the ports are closed. In State 3 (right side) Port 4 connects to the exhaust and port 2 connects to the supply. This valve could drive a 2 chamber cylinder by connecting port 2 to a chamber and port 4 to the other chamber.