Matrix Algebra

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1 Trace

- $\text{trace}(a + b) = \text{trace}(a) + \text{trace}(b)$
- $\text{trace}(a) = \text{trace}(a^T)$
- $\text{trace}(a^T r b^T) = a^T r b$ *confirm r does not need rotation*
- If a is $m \times n$ and b is $n \times m$ then $\text{trace}(ab) = \text{trace}(ba) = \text{trace}(a^T b^T)$

2 Kronecker and Vec

Definition: Kronecker product

$$a \otimes b = \begin{pmatrix} a_{11}b & \cdots & a_{1n}b \\ \vdots & \ddots & \vdots \\ a_{m1}b & \cdots & a_{mn}b \end{pmatrix} \tag{1}$$

Definition: Vec operator

$$\text{vec}[a] = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ a_{12} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \tag{2}$$

2.1 Properties

1. $$a \otimes b \otimes c = (a \otimes b) \otimes c = a \otimes (b \otimes c) \tag{3}$$
   provided the dimensions of the matrices allows for all the expressions to exist.

2. $$(a + b) \otimes (c + d) = a \otimes c + a \otimes d + b \otimes c + b \otimes d \tag{4}$$

3. $$(a \otimes b)(c \otimes d) = (ac) \otimes (bd) \tag{5}$$

4. $$(a \otimes b)(b \otimes d) = (ac) \otimes (bd) \tag{6}$$

5. $$(a \otimes b)^T = a^T \otimes b^T \tag{7}$$

6. $$(a \otimes b)^{-1} = a^{-1} \otimes b^{-1} \tag{8}$$

7. $$\text{vec}\left[ab^T\right] = b \otimes a \tag{9}$$

8. $$\text{vec}\,[abc] = (c^T \otimes a)\,\text{vec}\,[b] \tag{10}$$

9. If $\{\lambda_i, u_i\}$ are eigenvalues/eigenvectors of a and $\{\delta_i, v_i\}$ are eigenvalues/eigenvectors of b then $\{\lambda_i\delta_j, u_i \otimes v_j\}$ are eigenvalues/eigenvectors ofr $a \otimes b$

10. $$\det[a \otimes b] = \det[a]^m \det[b]^n \tag{11}$$

    where a, b are of order $n \times n$ and $m \times m$ respectively.

11. $$\text{trace}(a \otimes b) = \text{trace}[a]\,\text{trace}[b] \tag{12}$$

History

- The first version of this document was written on May 2004.

References