CSE 12:
Basic data structures and object-oriented design

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Lecture Eighteen
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More on sorting.

Insertion sort.

Insertion sort

- Like selection sort, **insertion sort** maintains a “sorted part” *S* and “unsorted part” *U* of the input array.
- With insertion sort, *S* is to the *left* of *U*.

Sorted part Unsorted part

- Insertion sort operates by repeatedly removing the *leftmost* element of *U* and *inserting* it into its “proper place” in *S*.

Insertion sort

- Example:

Sorted part

Unsorted part

6 1 4 3 8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5
1 6	4 3 8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5
1 6	4 3 8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5
1 6	4 3 8 7 2 5
1 4 6	3 8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5
1 6	4 3 8 7 2 5
1 4 6	3 8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	`6 1 4 3 8 7 2 5`
`6`	`1 4 3 8 7 2 5`
`1 6`	`4 3 8 7 2 5`
`1 4 6`	`3 8 7 2 5`
`1 3 4 6`	`8 7 2 5`

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5
1 6	4 3 8 7 2 5
1 4 6	3 8 7 2 5
1 3 4 6	8 7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5
1 6	4 3 8 7 2 5
1 4 6	3 8 7 2 5
1 3 4 6	8 7 2 5
1 3 4 6 8	7 2 5

Insertion sort

- Example:

Sorted part	Unsorted part
	6 1 4 3 8 7 2 5
6	1 4 3 8 7 2 5
1 6	4 3 8 7 2 5
1 4 6	3 8 7 2 5
1 3 4 6	8 7 2 5
1 3 4 6 8	7 2 5
1 3 4 6 7 8	2 5
1 2 3 4 6 7 8	5
1 2 3 4 5 6 7 8	Done.

Insertion sort

- With insertion sort, most of the "effort" is in *inserting* the element into its proper slot.
  - In contrast, with *selection sort*, most of the effort is in *finding* the largest element to insert.
- Like selection sort, insertion sort too can operate *in-place*:
  - When we remove the leftmost element *x* from *U*, we save it in a temporary variable.
  - To find *x*'s proper "slot": we "slide down" each element *y* of *S* to the right as long as $y > x$.
  - We finally insert *x*, and repeat until *U* is empty.

Insertion sort

- Example:

Sorted part Unsorted part

6 1 4 3 8 7 2 5 *x*: 6

Insertion sort

- Example:

Sorted part Unsorted part

`6 1 4 3 8 7 2 5`

Insertion sort

- Example:

Sorted part Unsorted part

`6 1 4 3 8 7 2 5`

x: 1

Insertion sort

- Example:

Sorted part Unsorted part

6 **4 3 8 7 2 5** *x*: 1

Move *y*=6 to the
right because *y*>*x*.

Insertion sort

- Example:

Sorted part Unsorted part

6 4 3 8 7 2 5 *x*: 1

Insertion sort

- Example:

Sorted part Unsorted part

`1 6 4 3 8 7 2 5`

Insertion sort

- Example:

Sorted part Unsorted part

`1 6 4 3 8 7 2 5` *x*: 4

Insertion sort

- Example:

Sorted part Unsorted part

`1 6` `3 8 7 2 5` x: 4

Move y=6 to the right because $y>x$.

Insertion sort

- Example:

Sorted part Unsorted part

`1 6 3 8 7 2 5`

x: 4

Insertion sort

- Example:

Sorted part Unsorted part

`1 4 6 3 8 7 2 5`

Insertion sort

- Example:

Sorted part Unsorted part

`1 4 6 3 8 7 2 5`

x: 3

Insertion sort

- Example:

Sorted part　　Unsorted part

1 4 6　　8 7 2 5　　*x*: 3

Move *y*=6 to the
right because *y*>*x*.

Insertion sort

- Example:

Sorted part Unsorted part

`1 4 6 8 7 2 5` *x*: 3

Insertion sort

- Example:

Sorted part Unsorted part

1 4 6 8 7 2 5 *x*: 3

Move *y*=4 to the
right because *y*>*x*.

Insertion sort

- Example:

Sorted part Unsorted part

`1    4  6  8  7  2  5`   *x*: 3

Insertion sort

- Example:

Sorted part Unsorted part

`1 3 4 6 8 7 2 5`

Insertion sort

- Example:

Sorted part Unsorted part

`1 3 4 6 8 7 2 5`

Insertion sort

- Example:

Sorted part Unsorted part

`1 3 4 6 7 8 2 5`

Insertion sort

- Example:

Sorted part Unsorted part

`1 2 3 4 6 7 8 5`

Insertion sort

- Example:

Sorted part Unsorted part

`1 2 3 4 5 6 7 8`

Insertion sort

The reason we need this variable is that "sliding" *y* to the right may overwrite the leftmost element of *U*.

- Pseudocode:

```
While U is not empty:
  Save leftmost element x of U into temporary variable.
  Remove x from U.
  Loop from right to left on element y of S:
    If y > x:
      Slide y to the right by one slot.
      Let y be the next-rightmost element of S.
    Else (y ≤ x):
      Insert x to the right of y.
```

Since the algorithm requires only $O(1)$ additional memory (to store x), it is still considered to operate "in-place".

Stability

- Insertion sort is *stable* as long as we "shift over" an element *y* in S if *y* > *x*.

Sorted part   Unsorted part

1 2 3_1 3_2 7 8

1 2 3_1 3_2 7 8

1 2 3_1 3_2 7 8

1 2 3_1 3_2 7 8

1 2 3_1 7 8   *x*: 3_2

We don't move $y=3_1$ to the right because *y not* > *x*.

Stability

- Insertion sort is *stable* as long as we "shift over" an element *y* in S if $y > x$.

Sorted part   Unsorted part

1 2 3_1 3_2 7 8

1 2 3_1 3_2 7 8

1 2 3_1 3_2 7 8

1 2 3_1 3_2 7 8

1 2 3_1 3_2 7 8

Stability

- Insertion sort is *stable* as long as we "shift over" an element *y* in S if $y > x$.

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$ Stable.

Stability

- If instead we “shift over” *y* whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

1 2 3_1 3_2 7 8
1 2 3_1 3_2 7 8
1 2 3_1 3_2 7 8
1 2 3_1 3_2 7 8
1 2 3_1 7 8 x: 3_2

We move $y=3_1$ to the right
because $y \geq x$.

Stability

- If instead we "shift over" *y* whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ \quad 3_1\ 7\ 8$ x: 3_2

Stability

- If instead we "shift over" *y* whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$

Stability

- If instead we "shift over" y whenever $y \geq x$, then insertion sort is *not* stable.

Sorted part Unsorted part

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_1\ 3_2\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$

$1\ 2\ 3_2\ 3_1\ 7\ 8$ Not stable.

Time cost analysis

- *Worst case*:
  - Outer loop executes *n* times.
  - Inner loop has to move all the elements of *S* to the right by one slot before inserting *x*.
    - Since *S* grows in size as outer loop iterates, this results in 1, 2, 3, ..., n-1 operations.
  - $1 + 2 + 3 + \ldots + n-1 = n(n-1)/2 = O(n^2)$.

Time cost analysis

- *Best case*:
  - Outer loop executes n times.
  - Inner loop only executes *once* -- x is inserted as the rightmost element of S.
    - This results in only 1 operation per outer loop iteration.
  - $1 + 1 + 1 + \ldots + 1 = O(n)$.
  - The *best case* is realized when the data are *already sorted*.

Heapsort.

Heapsort

- The *heap* data structure we covered earlier in the course turns out to be useful for sorting.
- A *heap* allows the removal of the *largest element* in $O(\log n)$ time.
- To see how this is useful in sorting, recall how *selection sort* operates:

```
While U is not empty:
  Remove the largest element of U and add it to S.
```

Heapsort

- Selection sort uses a simple *linear search* through *U* to find the largest element in *O*(*n*) time.
- Using a heap, we can do this in *O*(log *n*) time.
- This results in the following **heapsort** algorithm:

```
Build a heap from the data in U.
While U is not empty:
  Remove largest from U and add it to S.
```

Heapsort

- Building a heap from n data in *U* takes time at most O(n log n). *
- The loop iterates n times.
    - Finding+removing largest takes time $O(\log n)$.
- In total, heapsort takes time $O(n \log n) + n^*O(\log n) = O(n \log n)$ in both the *worst case* and *best case*.

* It's actually possible to heapify an array of n elements in $O(n)$ time, but that doesn't affect heapsort's *asymptotic* performance.

Heapsort

- Example:

Unsorted part

Sorted part

6 1 4 3 8 7 2 5

First, convert this into an array-based max-heap.

Heapsort

- Example:

Unsorted part

Sorted part

```
6 1 4 3 8 7 2 5
8 6 7 5 3 4 2 1
```

Heapsort

- Example:

Unsorted part

Sorted part

6 1 4 3 8 7 2 5

8 6 7 5 3 4 2 1

Now, repeatedly call `removeLargest()` and add that element to the sorted part.

Heapsort

- Example:

Unsorted part	Sorted part
6 1 4 3 8 7 2 5	
8 6 7 5 3 4 2 1	
7 6 4 5 3 1 2	8
6 5 4 2 3 1	7 8
5 3 4 2 1	6 7 8
4 3 1 2	5 6 7 8
3 2 1	4 5 6 7 8
2 1	3 4 5 6 7 8
1	2 3 4 5 6 7 8
Done.	1 2 3 4 5 6 7 8

Heapsort

- As with the other sorting algorithms we've examined, heapsort too can operate in-place.
- The "trick" to making it work is that the *input array* to heapsort will serve as the heap's *underlying storage*.
- Recall how an array-based heap is implemented internally:

```
int _numNodes;
int[] _nodeArray;  // Length >= _numNodes
```

Heapsort

- Whenever we add a new element to a heap, we store it in `_nodeArray[_numNodes]` and then increment `_numNodes`, e.g.:

`_nodeArray`

3	2	1					

`_numNodes: 3`

Before adding 6.

`_nodeArray`

3	2	1	6				

`_numNodes: 4`

After adding 6.

Heapsort

- We must then call **`bubbleUp`** on the new element.

`_nodeArray`

3	2	1	6				

Before **`bubbleUp`**.

`_numNodes: 4`

`_nodeArray`

3	6	1	2				

`_numNodes: 4`

`_nodeArray`

6	3	1	2				

After **`bubbleUp`**.

`_numNodes: 4`

Heapsort

- To make heapsort work in place, the heap we create will *use the input array as its underlying storage*.
  - No need to "insert" the elements to the array -- *they're already there*.
- Hence, to add an element to the heap, all we must do is:

1. Increment **`_numNodes`**
2. Call **`bubbleUp`** on the last element of **`_nodeArray`**.

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

  `6 1 4 3 8 7 2 5`

- The array above *is* the heap's underlying storage (**`_nodeArray`**). *
- Initially, **`_numNodes`** = 0.
- Each time we "add" an element to the heap, **`_numNodes`** will increase by 1.

* In practice, this requires adding another constructor to **`HeapImpl12`** that takes a single argument, **`int[] _underlyingStorage`**, passed in by the user.

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

6 1 4 3 8 7 2 5

`_numNodes: 0`

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 1 4 3 8 7 2 5`

`_numNodes:` `1`

Increment **`_numNodes`**

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 1 4 3 8 7 2 5`

`_numNodes: 1`

Call **`bubbleUp`**

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 1 4 3 8 7 2 5`

`_numNodes:` `2`

Increment **`_numNodes`**

Heapsort

- Example -- let’s turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 1 4 3 8 7 2 5`

`_numNodes: 2`

Call **`bubbleUp`**

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 1 4 3 8 7 2 5`

`_numNodes: 3`

Increment `_numNodes`

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 1 4 3 8 7 2 5`

`_numNodes: 3`

Call `bubbleUp`

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

6 1 4 3 **8 7 2 5**

`_numNodes:` **`4`**

Increment **`_numNodes`**

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 1 4 3 8 7 2 5`

`_numNodes: 4`

Call `bubbleUp`

Heapsort

- Example -- let’s turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 3 4 1` **`8 7 2 5`**

`_numNodes: 4`

Call **`bubbleUp`**

Heapsort

- Example -- let’s turn the following 8-element input array into a *heap*.

Heapified elements Non-heapified elements

`6 3 4 1 8 7 2 5`

`_numNodes:` `5`

Increment **`_numNodes`**

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements   Non-heapified elements

`6 3 4 1 8 7 2 5`

`_numNodes: 5`

Call `bubbleUp`

Heapsort

- Example -- let's turn the following 8-element input array into a *heap*.

Heapified elements   Non-heapified elements

`8 6 4 1 3` **`7 2 5`**

`_numNodes: 5`

Call **`bubbleUp`**

Keep repeating this process...

Heapsort

- Example -- let’s turn the following 8-element input array into a *heap*.

Heapified elements   Non-heapified elements

`8 6 7 5 3 4 2 1`

`_numNodes: 8`

Done.

- We have now constructed a heap *within the input array itself.*
  - This requires 0 extra storage.

Heapsort

- However, we're still not done.
- We still have to call `removeLargest()` repeatedly, and store its result into the *leftmost* position of the *sorted part* of the array.
- Since we're operating *in-place*, this will require that store the largest value *x* in a temporary variable.

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

8 6 7 5 3 4 2 1

`_max:`
`_numNodes: 8`

Save the heap's largest element in **`_max`**, and then remove the largest element.

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

8 6 7 5 3 4 2 1

`_max`: **8**

`_numNodes`: **8**

Save the heap's largest element in **`_max`**, and then remove the largest element.

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part Sorted part

`1 6 7 5 3 4 2`

`_max: 8`
`_numNodes: 7`

Calling `removeLargest` requires us to `trickleDown` from the root.

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part   Sorted part

`7 6 1 5 3 4 2`

Calling **`removeLargest`** requires us to **`trickleDown`** from the root.

`_max: 8`
`_numNodes: 7`

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part Sorted part

`7 6 4 5 3 1 2`

`_max: 8`
`_numNodes: 7`

Calling `removeLargest` requires us to `trickleDown` from the root.

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part Sorted part

7 6 4 5 3 1 2 8

Finally, we store **`_max`** into the *sorted part* of the array.

`_max:`

`_numNodes: 7`

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

`7 6 4 5 3 1 2 8`

`_max: 7`
`_numNodes: 7`

Save the heap's largest element in **`_max`**, and then remove the largest element.

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part

Sorted part

2 6 4 5 3 1

8

Calling `removeLargest` requires us to `trickleDown` from the root.

`_max: 7`

`_numNodes: 7`

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part Sorted part

6 2 4 5 3 1 8

Calling `removeLargest` requires us to `trickleDown` from the root.

`_max: 7`
`_numNodes: 7`

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call **`removeLargest()`** to populate the *sorted part* of the array:

Unsorted part Sorted part

`6 5 4 2 3 1` **`8`**

`_max: 7`
`_numNodes: 7`

Calling **`removeLargest`** requires us to **`trickleDown`** from the root.

Heapsort

- Given that the *unsorted part* of the array is now a valid *heap*, we can repeatedly call `removeLargest()` to populate the *sorted part* of the array:

Unsorted part Sorted part

6 5 4 2 3 1 7 8

_max:
_numNodes: 7

Finally, we store **`_max`** into the *sorted part* of the array.

Heapsort

- We repeatedly remove the largest element and store it into the leftmost slot of the unsorted part of the array, *until the heap is empty*.
- At that point, the array will be completely sorted.
- Since this required only one auxiliary variable (**`_max`**), the algorithm works *in-place*.

Heapsort

- In summary, heapsort is an in-place sorting algorithm whose best and worst case time costs are $O(n \log n)$.
- However, the algorithm is *not* stable because the heap ordering may cause the relative order of duplicate elements to become inverted.

Mergesort.

Approach 2: divide and conquer

- So far we’ve looked at sorting algorithms that partition the input array into a *sorted part* and *unsorted part*, and then “grow” the sorted part to be the entire array.
- An alternative approach altogether is based on the *divide-and-conquer* principle:
  - To sort a list of size *n*:
    - Divide the list into two halves (approx. size *n*/2).
    - Sort each half independently using recursion.
    - Combine the 2 sorted lists of *n*/2 elements into 1 sorted list of *n* elements.

Mergesort

- The first algorithm we examine that uses divide-and-conquer is **Mergesort**.
- Here’s the “main idea” behind the algorithm:
  - Suppose we have a *left list* and a *right list* that are *already sorted*.
  - To combine these two lists into one larger sorted list, we just:
    - Iterate through both lists simultaneously.
    - “Pick out” the smaller element from the current position of either the left or right list, and insert it into our *combined* list.

Merging two sorted lists

- Example:

Left list Right list Combined list

`1 3 4 6 2 5 7 8`

`i` **`j`** **`k`**

Iterate through both lists:

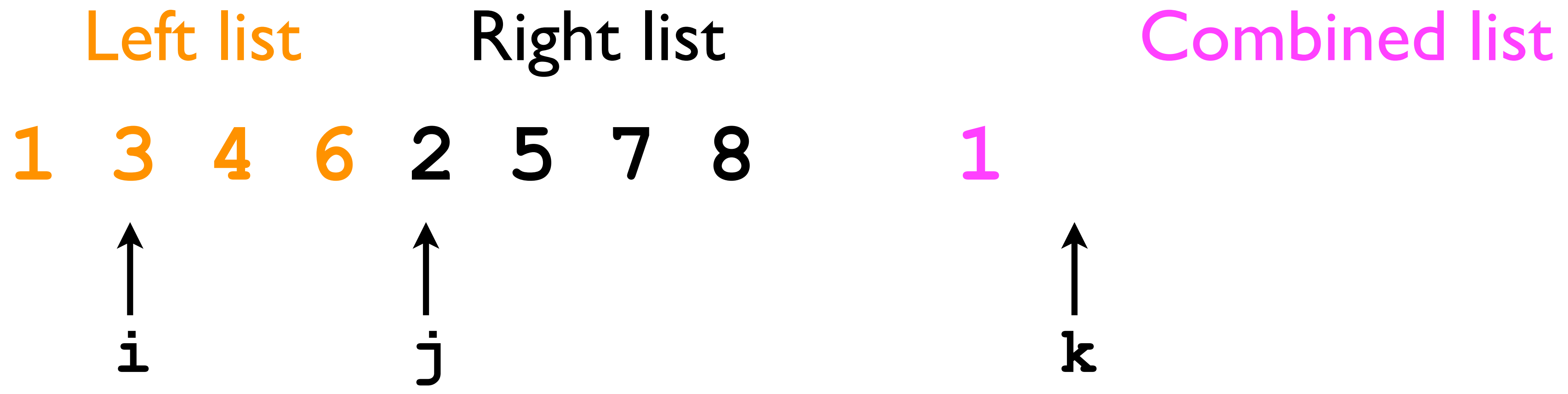
Pick out the smaller element *x* from the current position of either the left or right list;

Advance the pointer of whichever list contained x;

Then *insert x* into the combined list.

Merging two sorted lists

- Example:



Left list   Right list   Combined list

`1 3 4 6 2 5 7 8`   `1`

`i`   `j`   `k`

Iterate through both lists:

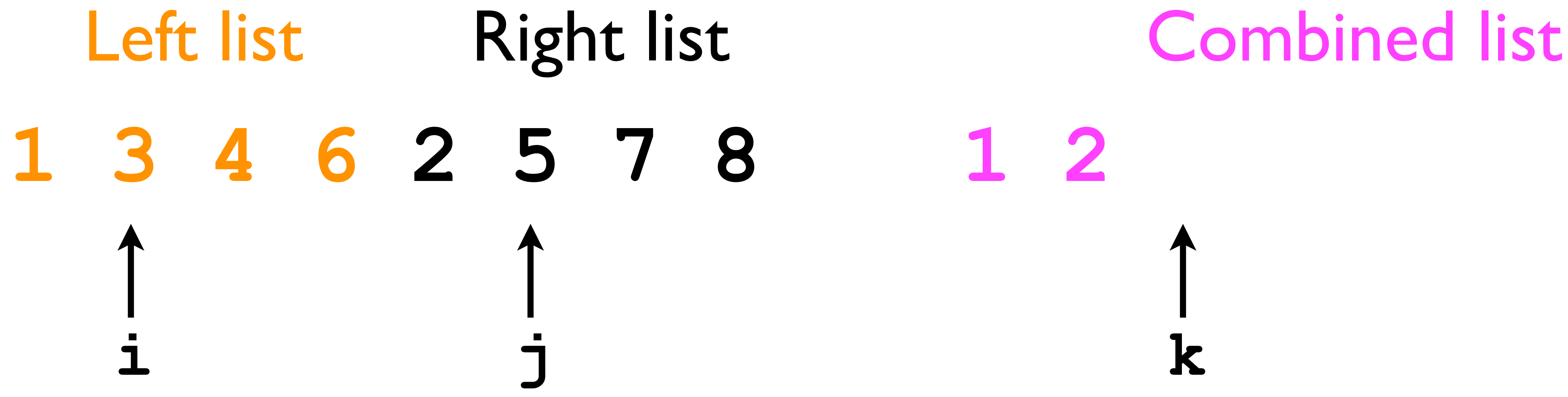
Pick out the smaller element *x* from the current position of either the left or right list;

Advance the pointer of whichever list contained x;

Then *insert x* into the combined list.

Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8` `1 2`

`i` `j` `k`

Iterate through both lists:

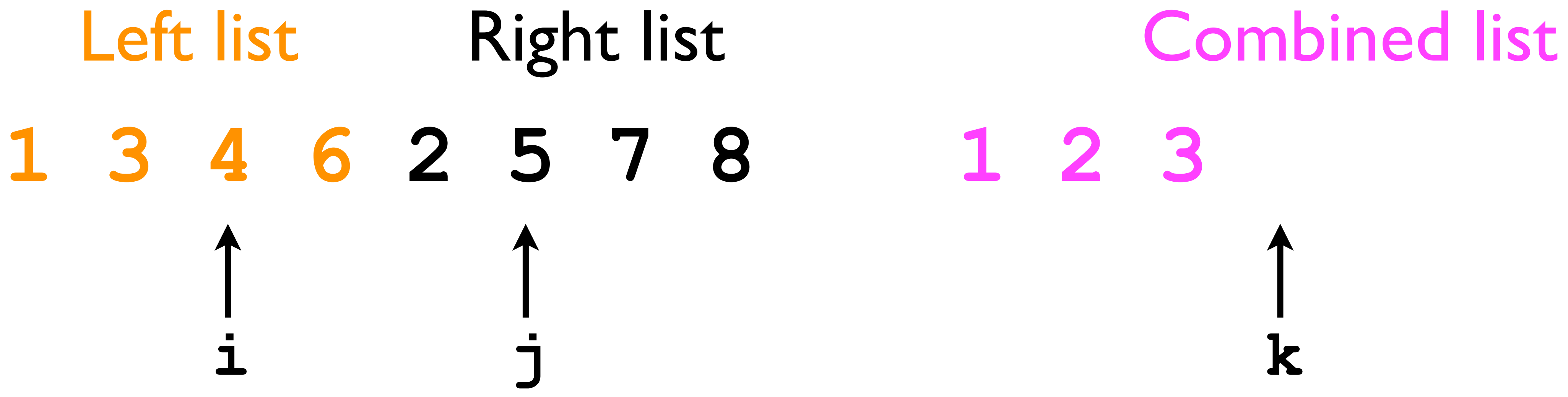
Pick out the smaller element *x* from the current position of either the left or right list;

Advance the pointer of whichever list contained x;

Then *insert x* into the combined list.

Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8` `1 2 3`

`i` `j` `k`

Iterate through both lists:

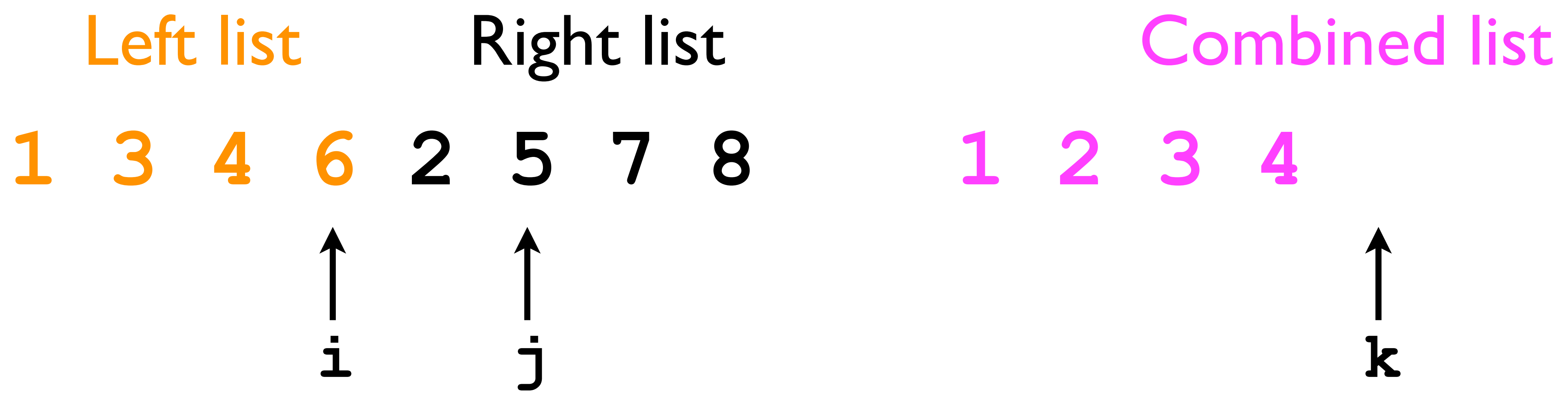
Pick out the smaller element *x* from the current position of either the left or right list;

Advance the pointer of whichever list contained x;

Then *insert x* into the combined list.

Merging two sorted lists

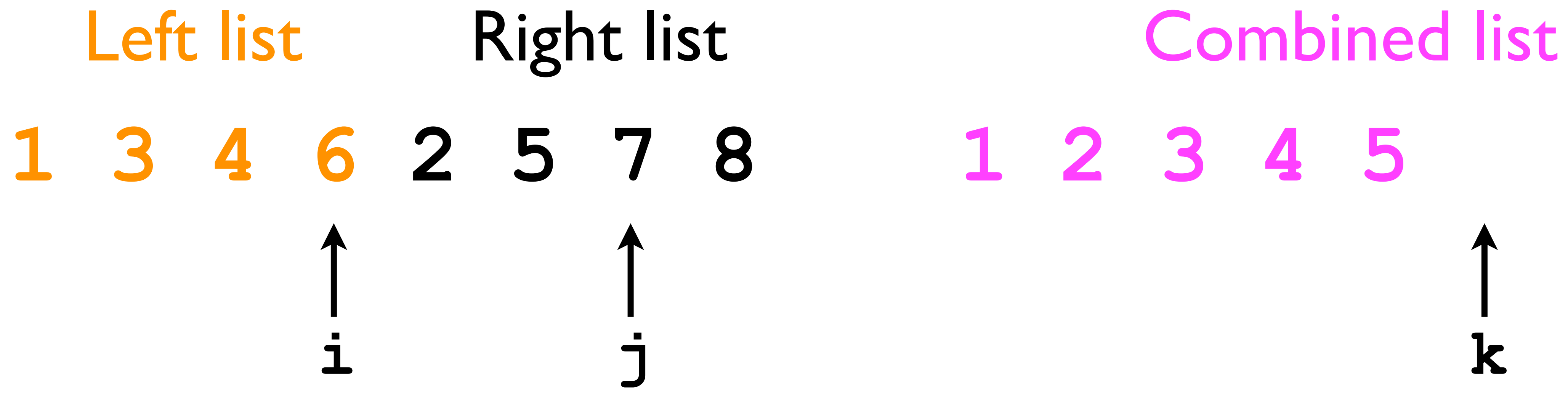
- Example:



Iterate through both lists:
Pick out the smaller element *x* from the current position
of either the left or right list;
Advance the pointer of whichever list contained x;
Then *insert x* into the combined list.

Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8` `1 2 3 4 5`

`i` `j` `k`

Iterate through both lists:

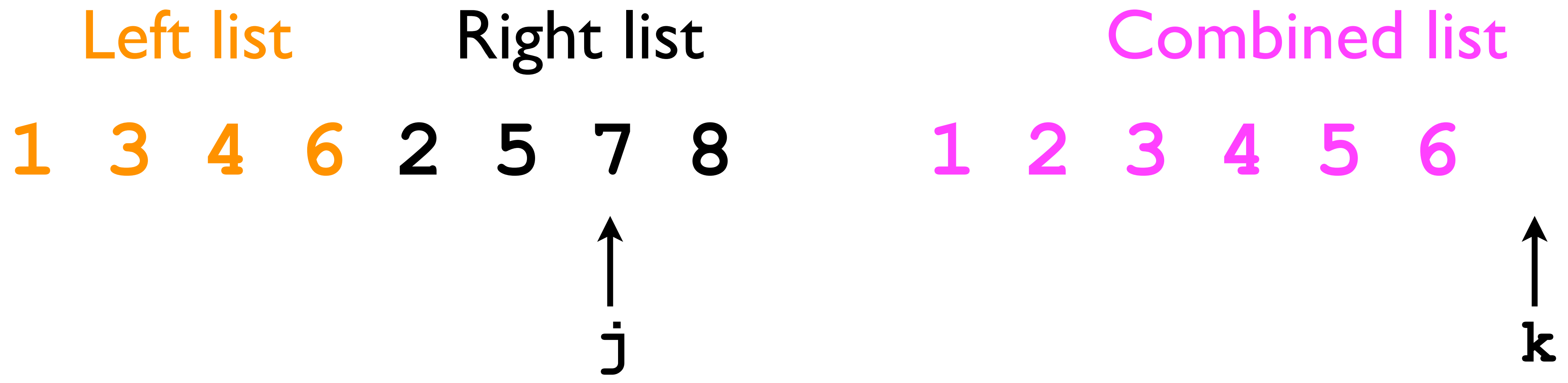
Pick out the smaller element *x* from the current position of either the left or right list;

Advance the pointer of whichever list contained x;

Then *insert x* into the combined list.

Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8` `1 2 3 4 5 6`

`j` `k`

Iterate through both lists:

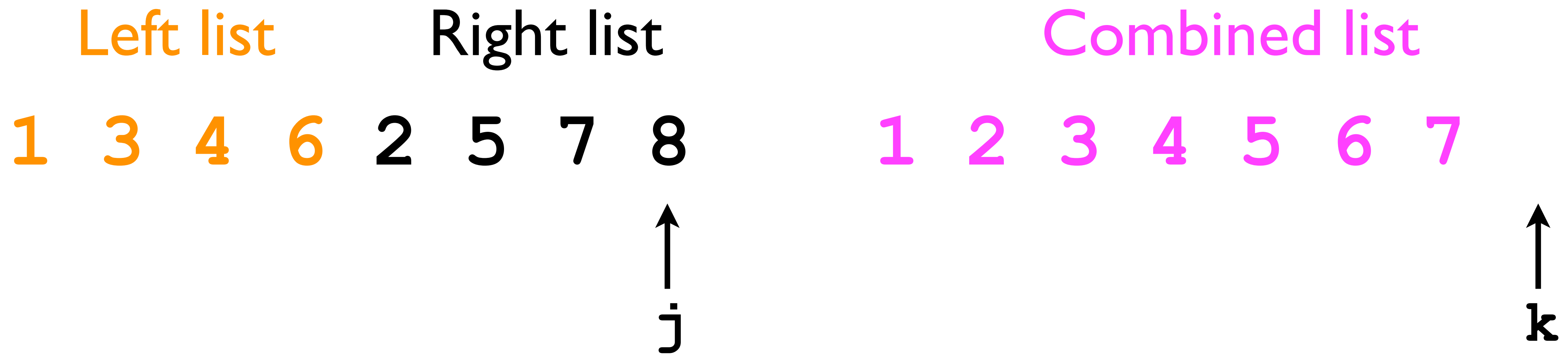
Pick out the smaller element *x* from the current position of either the left or right list;

Advance the pointer of whichever list contained x;

Then *insert x* into the combined list.

Merging two sorted lists

- Example:



Left list Right list Combined list

`1 3 4 6 2 5 7 8`

`1 2 3 4 5 6 7`

Iterate through both lists:

Pick out the smaller element *x* from the current position of either the left or right list;

Advance the pointer of whichever list contained x;

Then *insert x* into the combined list.

Merging two sorted lists

- Example:

Left list	Right list	Combined list
`1 3 4 6`	`2 5 7 8`	`1 2 3 4 5 6 7 8`

Done.

Iterate through both lists:
Pick out the smaller element *x* from the current position of either the left or right list;
Advance the pointer of whichever list contained x;
Then *insert x* into the combined list.

Mergesort

- Given a left list (*n*/2 elements) and a right list (*n*/2 elements), "merging" them into a combined list (*n* elements) takes time O(*n*).
- However, it requires that we allocate a *temporary array* of size *n*. *
  - Mergesort does not operate *in-place*.
- After merging, we copy the elements in the temporary array back into the input array.

* Except when using a linked-list representation.

Mergesort

- Given a procedure to *merge two sorted lists*, we can define a *recursive sorting algorithm* in the following way:
  - Given an input array:
    - If its length is 1, then it's already sorted.
    - Else:
      - Divide the list into two halves.
      - Recursively sort each half.
      - Merge their results into one combined list.

Mergesort

- Mergesort's pseudocode:

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Base case

Recursive part

- Let's see how it works in practice...

Mergesort

- Example:

First stage: recursively divide until we reach the base case.

6 1 4 3 8 7 2 5

Split list and recurse.

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Mergesort

- Example: First stage: recursively divide until we reach the base case.

6 1 4 3 8 7 2 5

6 1 4 3   8 7 2 5

Split list and recurse.

```
void mergesort (array) {
  If array.length == 1, then do nothing.
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    Split array evenly into leftArray and rightArray.
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}
```

Mergesort

- Example: First stage: recursively divide until we reach the base case.

6 1 4 3 8 7 2 5 — Split list and recurse.

6 1 4 3   8 7 2 5 — Split list and recurse.

6 1   4 3   8 7   2 5 — Split list and recurse.

6   1   4   3   8   7   2   5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Mergesort

- Example:

Each of these is a “list” (size 1) passed to a recursive call to Mergesort.

6 1 4 3 8 7 2 5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Mergesort

- Example:

Second stage: merge each pair of sorted sub-lists.

```
1 6     3 4     7 8     2 5
```

Merge the two sub-lists.

```
6   1   4   3   8   7   2   5
```

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Mergesort

- Example:

Second stage: merge each pair of sorted sub-lists.

1 3 4 6 2 5 7 8

Merge the two sub-lists.

1 6 3 4 7 8 2 5

6 1 4 3 8 7 2 5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Mergesort

- Example:

Second stage: merge each pair of sorted sub-lists.

1 2 3 4 5 6 7 8

Merge the two sub-lists.

1 3 4 6 2 5 7 8

1 6 3 4 7 8 2 5

6 1 4 3 8 7 2 5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Mergesort

- Example:

Done.

1 2 3 4 5 6 7 8

1 3 4 6 2 5 7 8

1 6 3 4 7 8 2 5

6 1 4 3 8 7 2 5

```
void mergesort (array) {
  If array.length == 1, then do nothing.
  Else:
    Split array evenly into leftArray and rightArray.
    mergesort(leftArray);
    mergesort(rightArray);
    Merge the leftArray and rightArray into array
}
```

Mergesort

- Example:

```
          1 2 3 4 5 6 7 8

      1 3 4 6         2 5 7 8

    1 6    3 4     7 8    2 5

   6   1   4   3   8   7   2   5
```

- The *depth* of this recursive call stack is the *number of times we can divide* n *by 2*, i.e., *O*(log *n*).

Mergesort

- Example:

`1 2 3 4 5 6 7 8`

`1 3 4 6` `2 5 7 8`

`1 6` `3 4` `7 8` `2 5`

`6` `1` `4` `3` `8` `7` `2` `5`

- At each level, *each element* in the input array had to be "touched" *once* (for the `merge` operation).
- *In total*: $O(\log n) * n = O(n \log n)$.

Mergesort

- Because Mergesort's dividing and merging requires the same number of operations *regardless* of the particular input, Mergesort's *best case* and *worst case* time complexities are both $O(n \log n)$.
- Mergesort is *stable* as long as the `merge` procedure selects the left array's x in the case of ties.

Quicksort.

Tomorrow.

Sorting demo.