CSE 12:

Basic data structures and object-oriented design

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Lecture Sixteen
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More on hash tables.

Hash tables

- In the previous lecture we discussed how *hash tables* enable *O*(1)-time **`add`**/**`find`**/**`remove`** operations in the average case.
    - The trade-off necessary to achieve *O*(1) time was the *extra space* needed to store a large, sparse array.
- Hash tables consist of a *large array*, plus a *hash function* to distribute the user's data "evenly" across the array.
    - The input to the hash function is the *key*, and its output is an *index* into the hash table's array.
    - Simple example:

```
int hashFunction (int key) {
  return key % M;  // M is size of _array
}
```

Keys and hash codes

- So far we have assumed that the *key* is always an *integer*, e.g., `studentID`.
- But what if wanted the student's `fullName` (i.e., a `String`) to be the key?
- Java gives us additional flexibility in how *keys* are converted into array indices.
    - Instead of hashing the key directly, we instead hash the key's **hash code**.
- A *hash code* is a way of describing any object `o` using just a primitive `int`.

Hash code examples

- Suppose our key is:
  - A single character `c`:
    - We could convert `c` into its ASCII value, which is an integer (from 0-127).
  - A `String s` of characters:
    - We could convert each `c` in `s` to its ASCII value, and then add them together.
  - An image `im`:
    - We could add together the pixel values across all three (R,G,B) channels.

Note: these are just hypothetical examples, not necessarily how Java actually implements hash codes!

Keys and hash codes

- The hash code serves as an "intermediary value" between the object's key and its assigned array index in a hash table.
- Instead of just

  ```
  _array[hashFunction(key)],
  ```

  `key` would have to be an integer.

  we instead write:

  ```
  _array[hashFunction(key.hashCode())];
  ```

  Now, `key` can be anything.

- The `Object.hashCode()` method converts any Java object into an *integer*.

hashCode()

- In Java, all objects support the **`hashCode()`** method, defined in class **`Object`**.
  - By default, **`hashCode()`** simply returns the object's location (address) in memory.
- A subclass **`A`** can override the default implementation when a customized implementation would improve performance, i.e., result in fewer *collisions*, *or* when **`A`** overrides the **`equals(o)`** method (more later).

hashCode()

- In Java, the **`hashCode()`** method *must* uphold two properties:

  1. *Deterministic* -- multiple subsequent calls to **`hashCode()`** on the *same object* o must return the same value.

     - Otherwise, **`hashFunction(key.hashCode())`** would map into a different array index -- and the hash table wouldn't be able to find o.

```
_array[hashFunction(o.key.hashCode())] = o;   // Add
...
return _array[hashFunction(o.key.hashCode()]; // Find
```

hashCode()

2. *Consistent across equal instances* -- if **o1.equals(o2)**, then **o1.hashCode()** *must* equal **o2.hashCode()**:

```
final String s1 = "hello";
final String s2 = new String("hello");  // Distinct copy
int hashCode1 = s1.hashCode();
int hashCode2 = s2.hashCode();  // Must equal hashCode1
```

- This means that if class **A** overrides the **equals()** method, then it must also override **hashCode()**.

- Calling **hashCode()** is sometimes faster than calling **equals(o)**; hence, **hashCode()** offers a "fast check" that objects o1 and o2 might be equal:

  - if **o1.hashCode() != o2.hashCode()**, then **o1** *cannot* equal **o2**.

`hashCode()`

- In addition, it is *desirable* for `hashCode()` to have:

3. *Wide distribution across instances* -- `hashCode()` should return *different* values for *different* instances of the same class as much as possible.

    - If `A.hashCode()` returned the *same* hash value for *every* instance `o`, then *all* objects of type `A` would map into the same array index. `hashCode()` is always the same.

```
_array[hashFunction(key1.hashCode())] = o1;
_array[hashFunction(key2.hashCode())] = o2;  // Collision
_array[hashFunction(key3.hashCode())] = o3;  // Collision
_array[hashFunction(key4.hashCode())] = o4;  // Collision
```

    - This would yield terrible (*O*(*n*)) hash performance!

`hashCode()` and `equals()`: Example 1

- The `String` class overrides the `equals()` method so that two distinct `String` objects `s1` and `s2` whose character sequences are identical are defined to be *equal*, e.g.:

```
String s1 = “test1”;
String s2 = new String(“test1”);  // distinct copy

boolean isSameAddress = (s1 == s2);  // false
boolean isEqual = s1.equals(s2);  // true
```

`hashCode()` and `equals()`: Example 1

- Since `s1` and `s2` are equal, their hash codes *must* be equal as well (according to `hashCode()` contract):

```
String s1 = “test1”;
String s2 = new String(“test1”);  // distinct copy

int hashCode1 = s1.hashCode();  // 110251487
int hashCode2 = s2.hashCode();  // 110251487
boolean isSameHashCode = (hashCode1 == hashCode2);  // true
```

`hashCode()` and **`equals()`**: Example 1

- The **`String.hashCode()`** method is implemented in the following way:
    - If the length *n* of **`s`** is 0, then **`s.hashCode()`** is 0.
    - Otherwise, **`s.hashCode()`** is:
        - $s[0] * 31^{n-1} + s[1] * 31^{n-2} + ... + s[n-1]$
- This formula ensures that **`String`**s with equal contents have the same hash code.
- It also tends to “spread” the hash codes of various **`String`**s evenly over the entire range of integers (-2^{31} to $+2^{31}-1$).

Hash table ADTs

- So far we've focused more on how a hash table is implemented *internally* and less how a user would *use* it.
- There are two different *interfaces* that a hash table ADT might offer.
- The interface varies depending on whether:

1. Key is a field *inside* the whole record.
2. Key is *separate* and stored *outside* the record.

Key inside the record

- In some previous examples we've conceptualized the *key* as a *field* within the whole object, e.g.:

```
class Student {
  int _studentID;
  String _firstName, _lastName;
  boolean _ownsTeddyBear;
}
```

- This implementation of *keys* then lends itself to the following hash table *interface*:

```
interface HashTable<T extends HasKey> {
  void add (T o);
  T get (T o);
}
```

where the hypothetical `HasKey` interface guarantees that `T` offers a method called `Object getKey()`.

Key inside the record

- The `add(o)` and `get(o)` methods might then be implemented as:

Here we're assuming that each `T` offers some method `getKey()` which returns the object's key -- e.g., the `_studentID` field in `Integer` form.

```
void add (T o) {
  final Object key = o.getKey();
  _array[hashFunction(key.hashCode())] = o;
}

T get (T o) {
 final Object key = o.getKey();
  return _array[hashFunction(key.hashCode())];
}
```

Key inside the record

- Since *every* Java object offers a **`hashCode()`** method, we can get rid of defining the key at all:

```
void add (T o) {
  _array[hashFunction(o.hashCode())] = o;
}
```

Now we just compute the hash code of o directly.

```
T get (T o) {
  return _array[hashFunction(o.hashCode())];
}
```

Key inside the record

- We can then simplify the interface of the hash table:

```
interface HashTable<T> {
  void add (T o);
  T get (T o);
}
```

No longer necessary for `T` to implement some `HasKey` interface.

- This is the interface used in P5.
  - Notice how the `add(o)` and `get(o)` methods are identical as for lists, BSTs, etc.

Key inside the record

- The user can then use the hash table as follows:

```
class Student {
  int _studentID;
  ...
  int hashCode () {
    return _studentID;
  }
}

final hashTable<Student> students =
  new HashTable<Student>();

students.add(new Student(
  12345, “Jacky”, “O’Nassis”, true
));
students.add(new Student(
  9231, “Bette”, “Midler”, false
));

...
final Student bette = students.get(new Student(9231));
```

She has a teddy bear.

She does not.

Key outside the record

- More commonly, however, hash tables *separate* the *key* from the *value*.
- A typical hash table interface might be:

```
interface HashTable<K,V> {
  void put (K key, V value);
  V get (K key);
}
```

Here, we are defining *two different* type parameters K (for keys) and V (for values).

Key outside the record

- The user would then use the hash table in the following way:

```
class Student {

  String _firstName, _lastName;
  boolean _hasTeddyBear;
}
```

No need for explicit `_studentID` field.

```
final HashTable<Integer,Student> hashTable =
  new HashTable<Integer,Student>();
hashTable.put(12345, new Student(
  “Jacky”, “O’Nassis”, true
));

...

final Student jacky = hashTable.get(12345);
```

Dictionaries

- Separating keys from values is especially useful when we use a hash table as a *dictionary*.
- A **dictionary** is a data structure for storing a set of associations between keys and values.
    - Each key can be associated with at most one value.

Dictionaries

- Examples:
  - We can create a dictionary of English words to their meanings:

```
HashTable<String,String> englishDictionary =
  new HashTable<String,String>();
englishDictionary.put(
  “eggplant”,
  “The somewhat large egg-shaped fruit of a
   tropical Old World plant, eaten as a vegetable.”
);

...

String meaning = englishDictionary.get(“eggplant”);
```

Caches.

Caches

- Having concluded our discussion of hash tables, we can now show a useful example of *combining* two data structures to build a third: in this case, a *cache*.
- Consider a situation in which a program needs to retrieve data from a container that is *slow*.
  - The slow speed might arise due to a long distance over which the data must travel, or to the slow data rate at which a device can deliver information.

Caches

- Examples:
  - A web browser downloads a webpage from an *external server*. Server is far away.
  - A spreadsheet program loads a file from *disk*. Disk is slow.
  - The CPU must read the value of a variable stored in *main memory* (instead of on-chip storage). RAM is slow.
- In each case, the program *fetches* data from *secondary storage* and loads it into *primary storage*.
  - Primary storage is faster and "closer" to the user than secondary storage.
  - What is "slow" in one context may be "fast" in another.

Caches

- Examples:
  - A web browser downloads a webpage from an *external server*.
    - Primary storage: computer memory (RAM) and/or disk.
    - Secondary storage: web server.
  - A spreadsheet program loads a file from *disk*.
    - Primary storage: computer memory (RAM).
    - Secondary storage: disk.
  - The CPU must read the value of a variable stored in *main memory* (instead of on-chip storage).
    - Primary storage: CPU registers.
    - Secondary storage: computer memory (RAM).

Caches

- Now, suppose that the *same* data X tends to be fetched from secondary storage *repeatedly*.
- In this case, we can save time by introducing an *intermediary* data container -- a *cache* -- that "remembers" the data fetched from secondary storage.
- A **cache** is a data structure that offers *high-speed* access to a *small* amount of data that must otherwise be written to/read from a *slower*, secondary storage container.

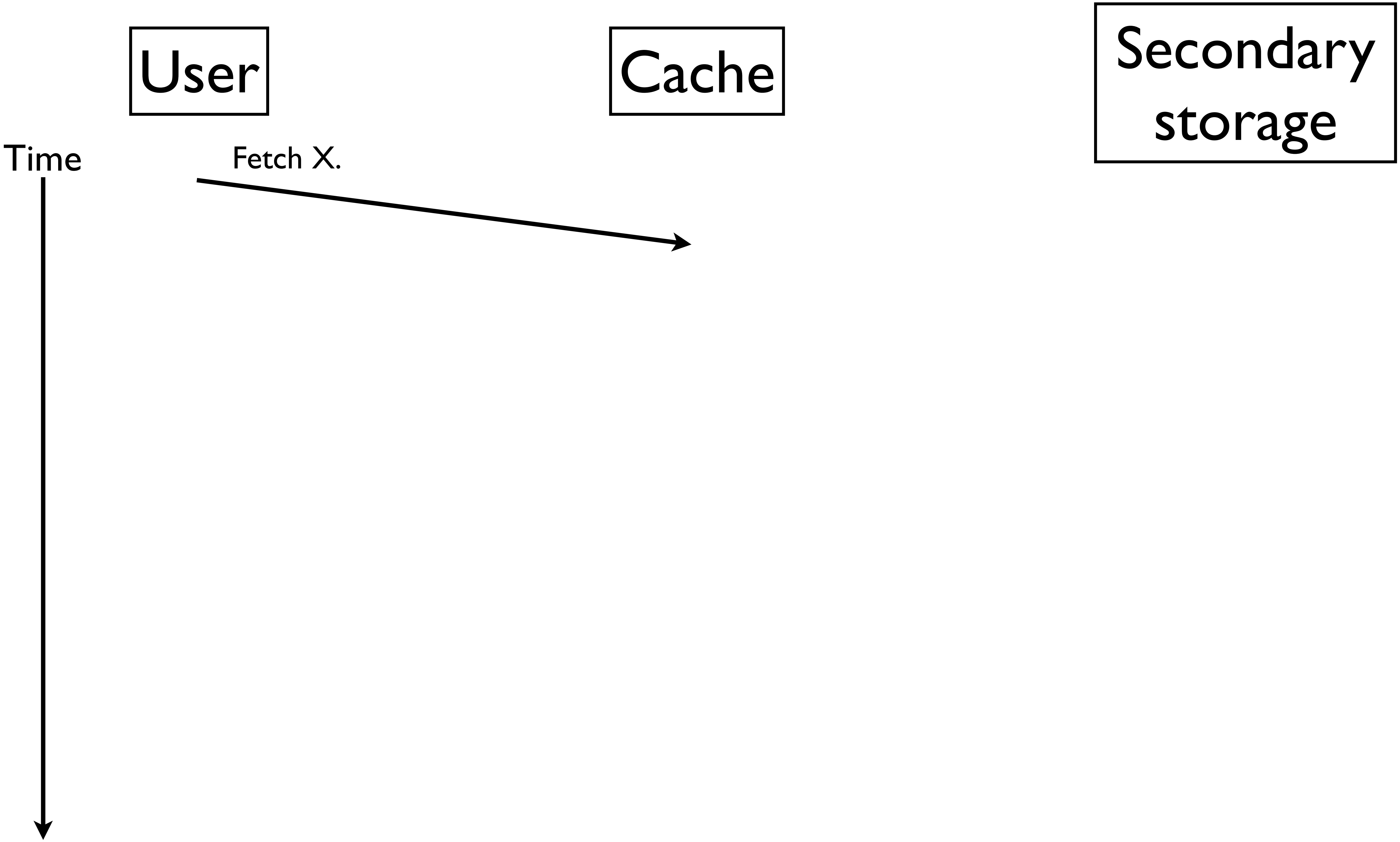
Caches: small and fast

- Caches are inherently *fast* and *small*:
    - *Fast* because they reside in primary storage, not secondary storage.
        - If they were slow, we'd forget the cache and just access secondary storage directly.
    - *Small* because they are typically more expensive than secondary storage.
        - If they were cheap, we'd just store *everything* in the cache and forget secondary storage.

Caches in action

- A user's request to fetch data X from secondary storage is "intercepted" by the cache:
    - If the cache already contains X, then the cache *returns* X to the user immediately.
        - Fetching X from secondary storage is unnecessary.
    - Otherwise (cache does not contain X), the cache *forwards* the user's request to secondary storage.
- Both *read* and *write* caches exist; here, we deal only with *read* caches.

Caches



User

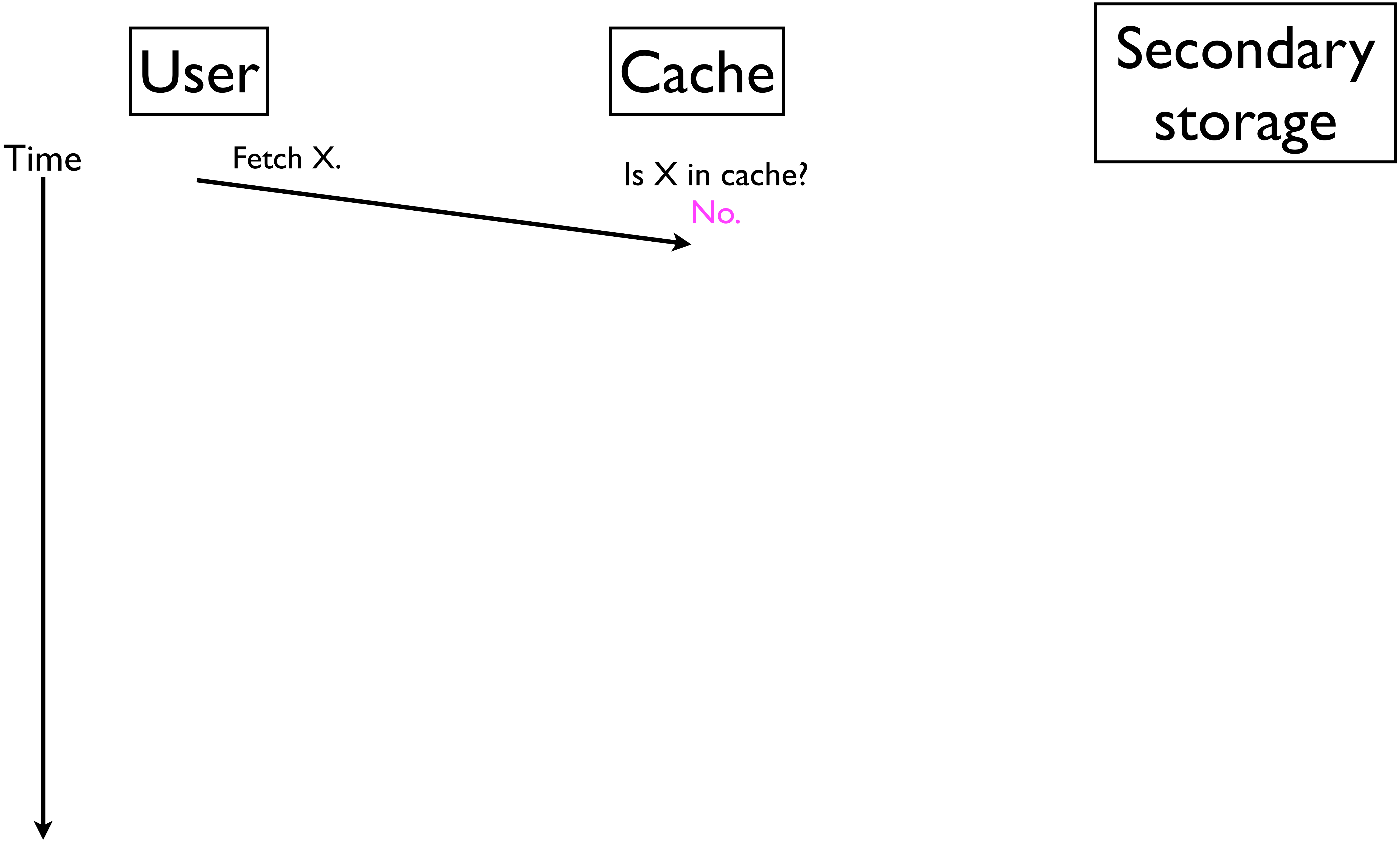
Cache

Secondary storage

Time

Fetch X.

Caches



User

Cache

Secondary storage

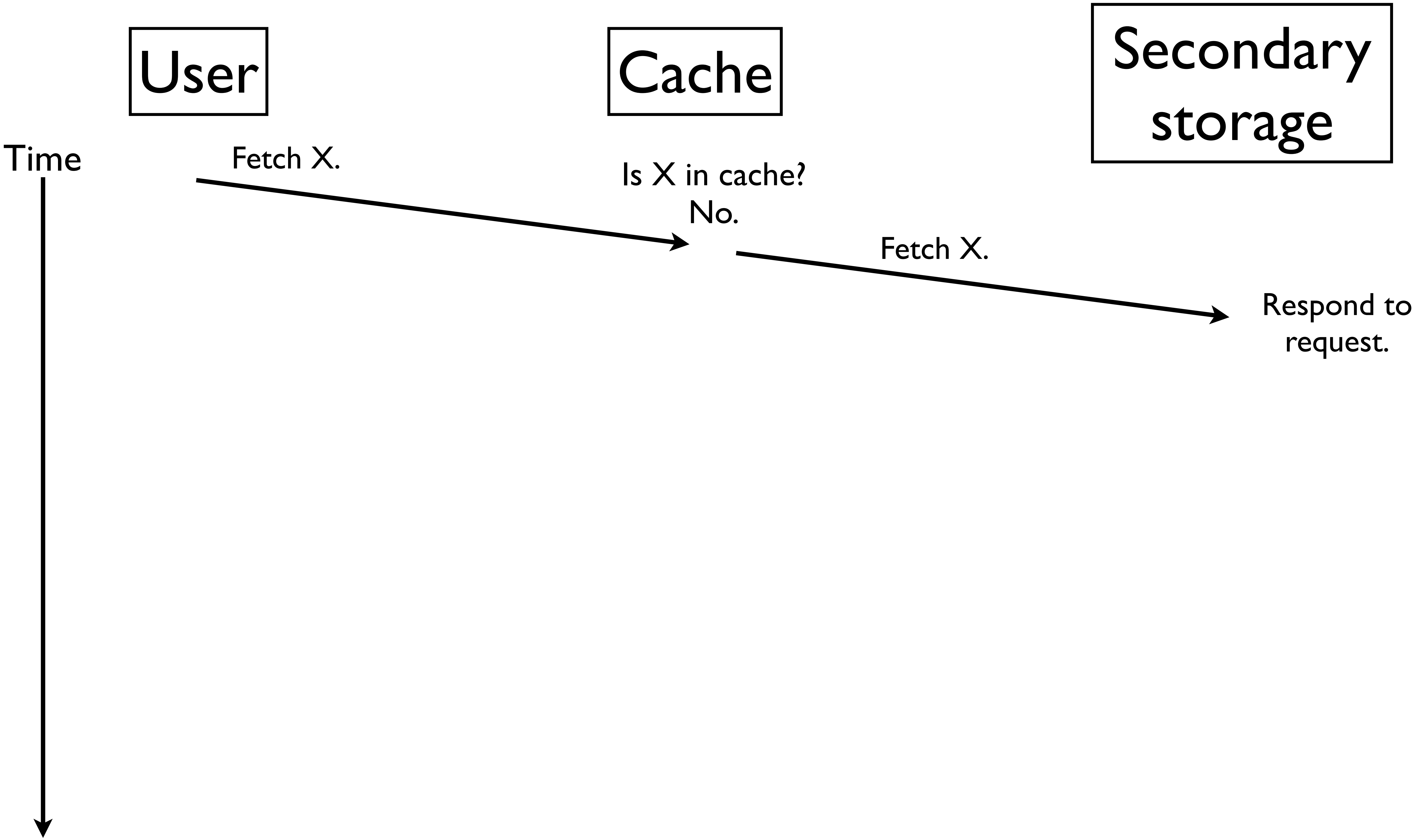
Time

Fetch X.

Is X in cache?

No.

Caches



User

Cache

Secondary storage

Time

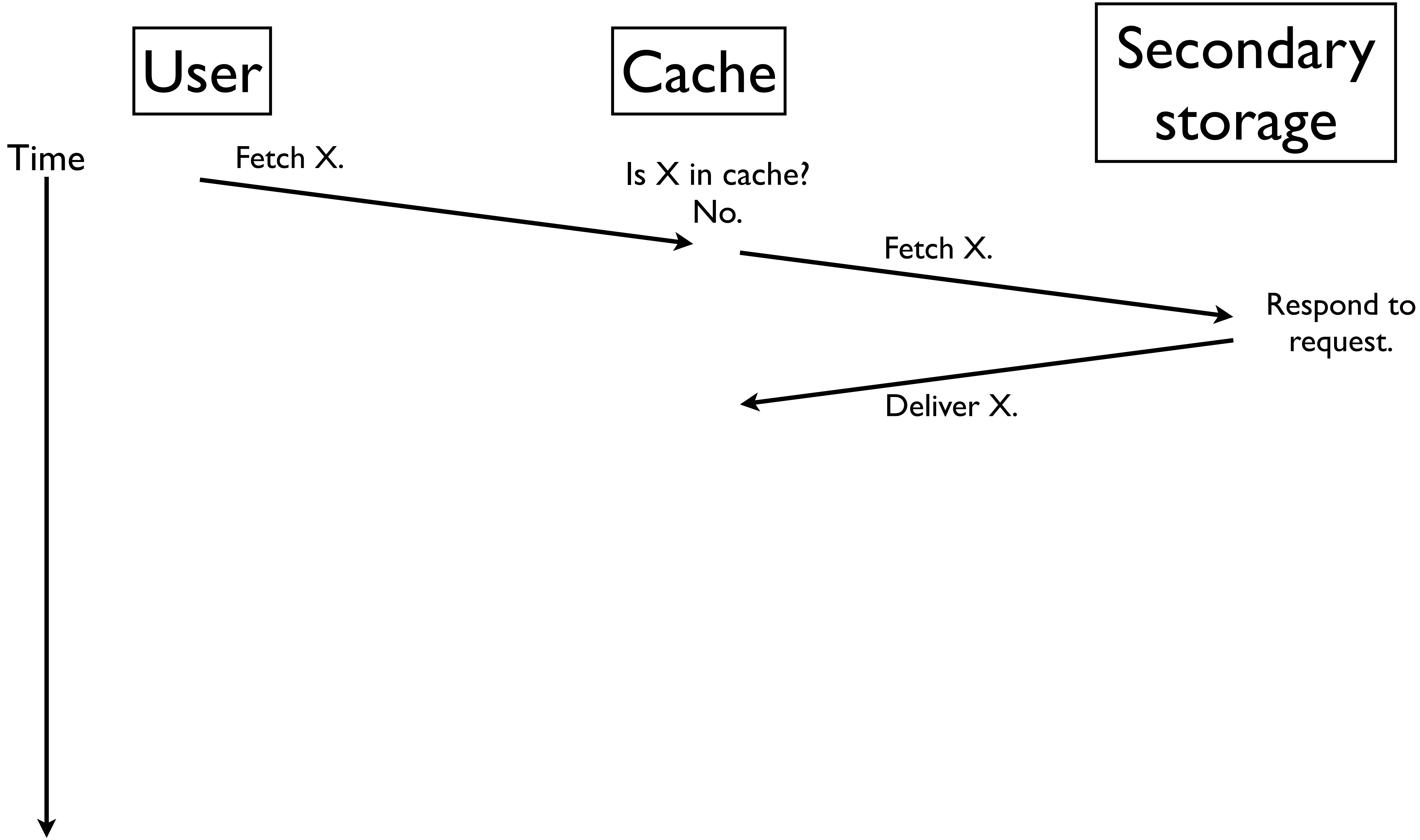
Fetch X.

Is X in cache?
No.

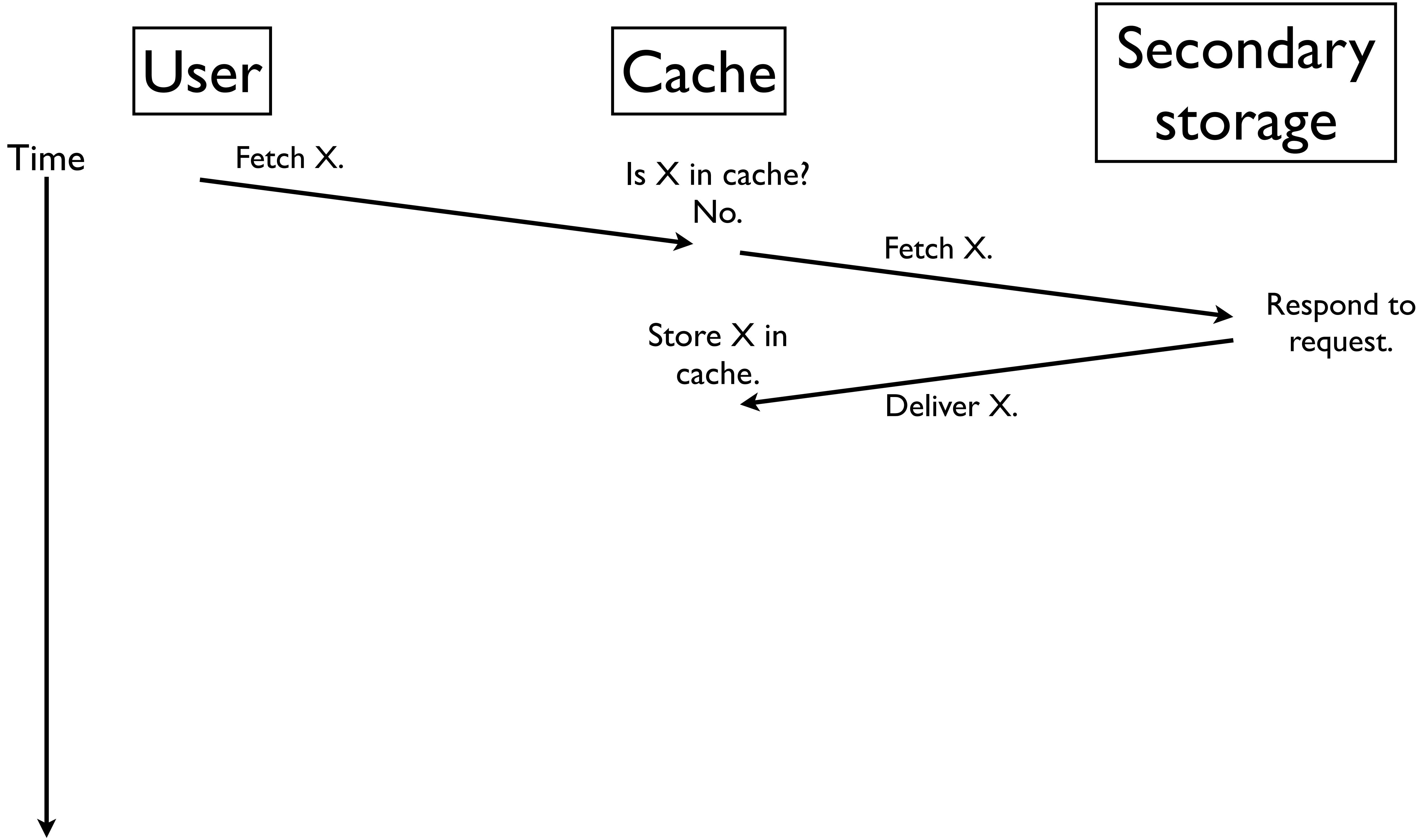
Fetch X.

Respond to request.

Caches


User
Cache
Secondary storage
Time
Fetch X.
Is X in cache?
No.
Fetch X.
Respond to request.
Deliver X.

Caches



User

Cache

Secondary storage

Time

Fetch X.

Is X in cache? No.

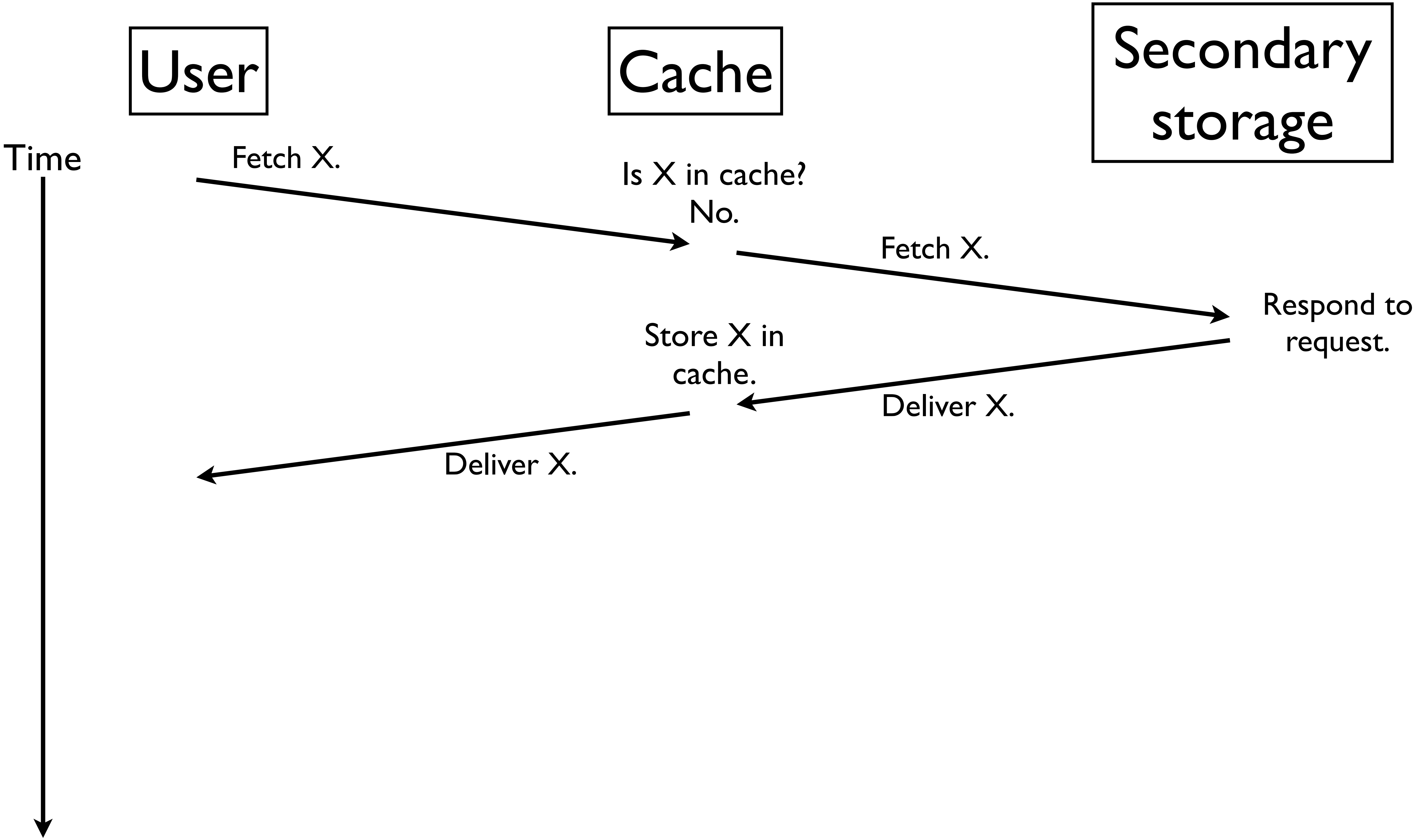
Fetch X.

Respond to request.

Store X in cache.

Deliver X.

Caches



User

Cache

Secondary storage

Time

Fetch X.

Is X in cache? No.

Fetch X.

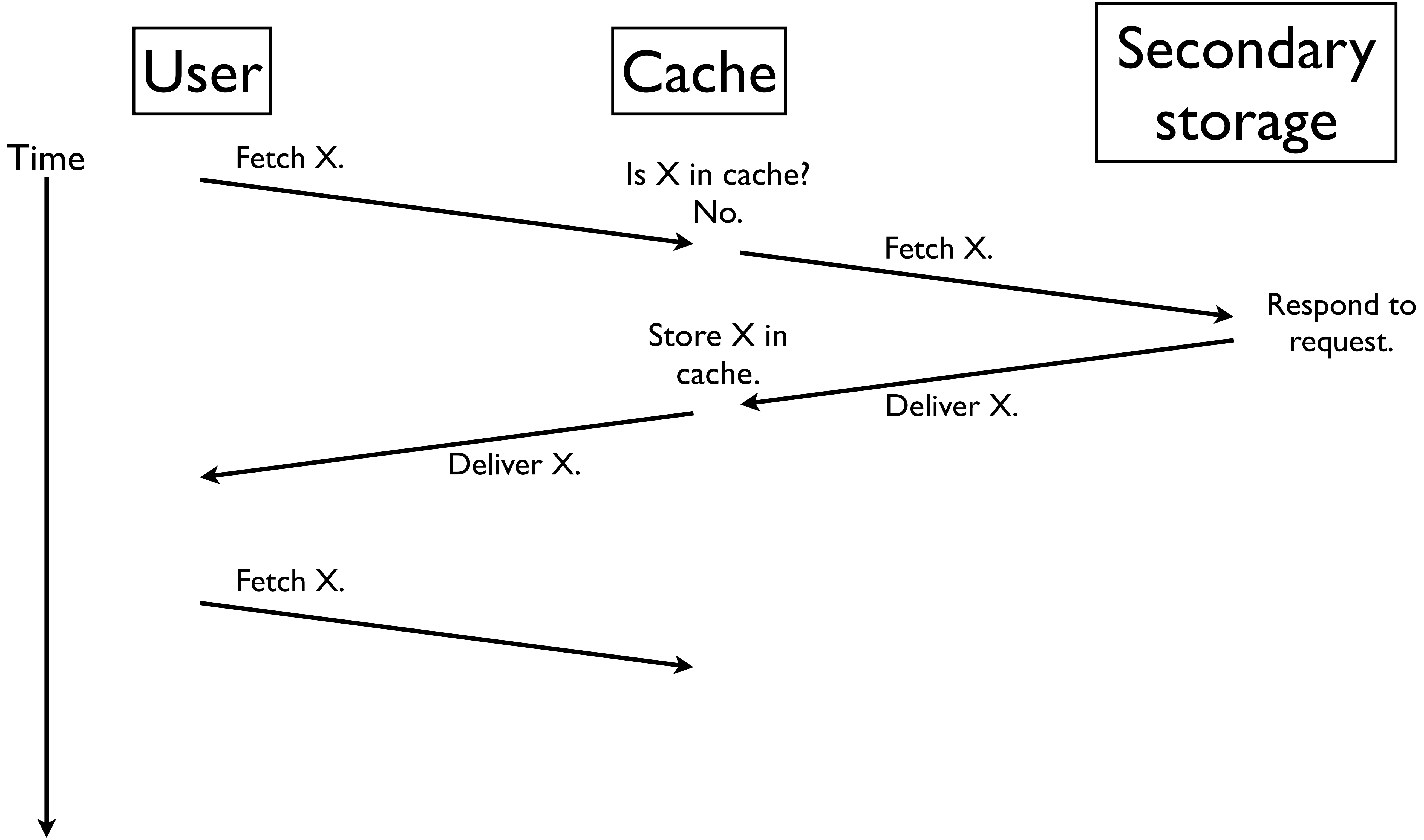
Respond to request.

Store X in cache.

Deliver X.

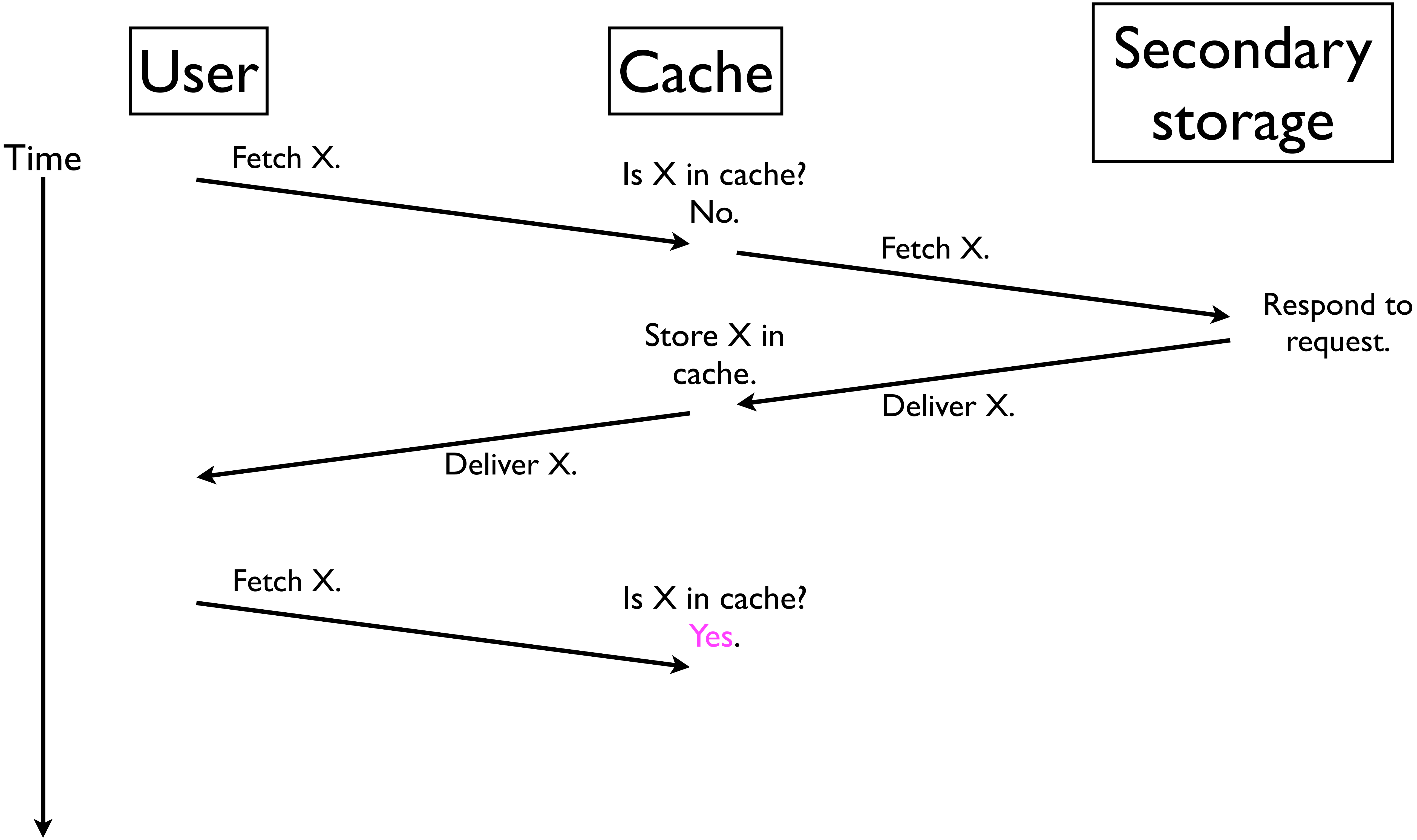
Deliver X.

Caches



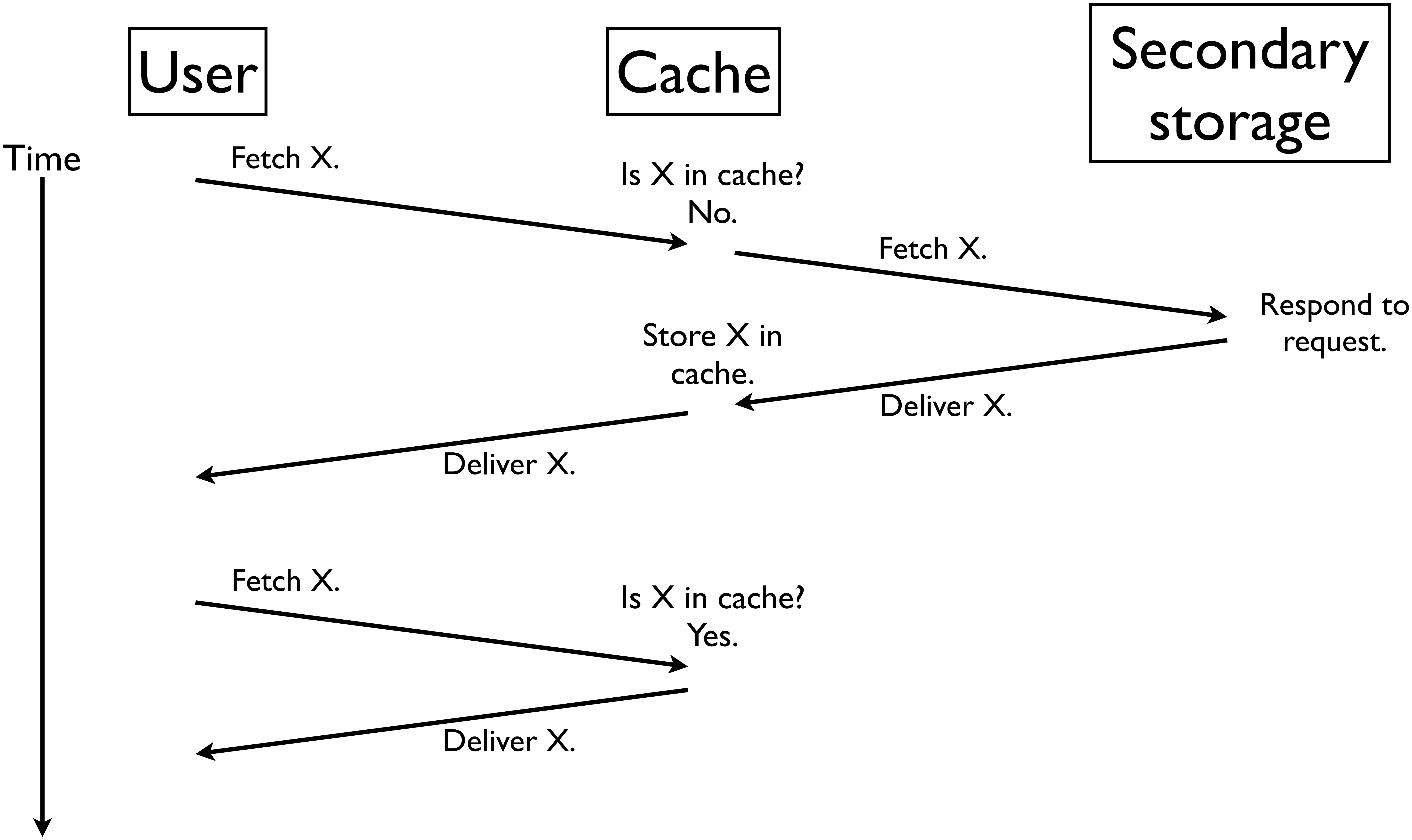
User
Cache
Secondary storage
Time
Fetch X.
Is X in cache? No.
Fetch X.
Respond to request.
Store X in cache.
Deliver X.
Deliver X.
Fetch X.

Caches



User
Cache
Secondary storage
Time
Fetch X.
Is X in cache?
No.
Fetch X.
Respond to request.
Store X in cache.
Deliver X.
Deliver X.
Fetch X.
Is X in cache?
Yes.

Caches



User

Cache

Secondary storage

Time

Fetch X.

Is X in cache? No.

Fetch X.

Respond to request.

Deliver X.

Store X in cache.

Deliver X.

Fetch X.

Is X in cache? Yes.

Deliver X.

Caches: definitions

- If the user requests item X from the cache, and X is contained in the cache, then we have a **cache hit**.
- Otherwise, if X is *not* in the cache, then we have a **cache miss**.
    - X must then be fetched from secondary storage.
- The size of the cache is always *finite*.
- For every cache miss: if the cache is *full*, the cache must decide which element to "forget", i.e., **evict**.
- The choice of which data to evict can affect the cache **miss rate** (fraction of cache accesses that miss) and thereby the performance of the computer system.

Eviction policies

- The algorithm that decides which object to evict is called an **eviction policy**.
- The choice of eviction policy can make a large impact on system performance.
- An *optimal* eviction policy determines which element o in the cache will not be used again for the longest period of time, and then evicts o.
  - This minimizes the expected cache miss rate.
- Unfortunately, this optimal policy is rarely achievable because it's difficult to predict which items will be needed in the future.

Least-recently-used caches

- One of the most commonly implemented eviction policies is *least-recently-used* (LRU).
- Whenever we must evict an element from the cache, we pick the least-recently-used element.
  - *Justification*: It seems reasonable that an item that has not been used in a long time will continue not to be requested for a while longer.
- Empirically, LRU has shown to perform “similarly” to the *optimal* eviction policy in many practical applications.

LRU in action

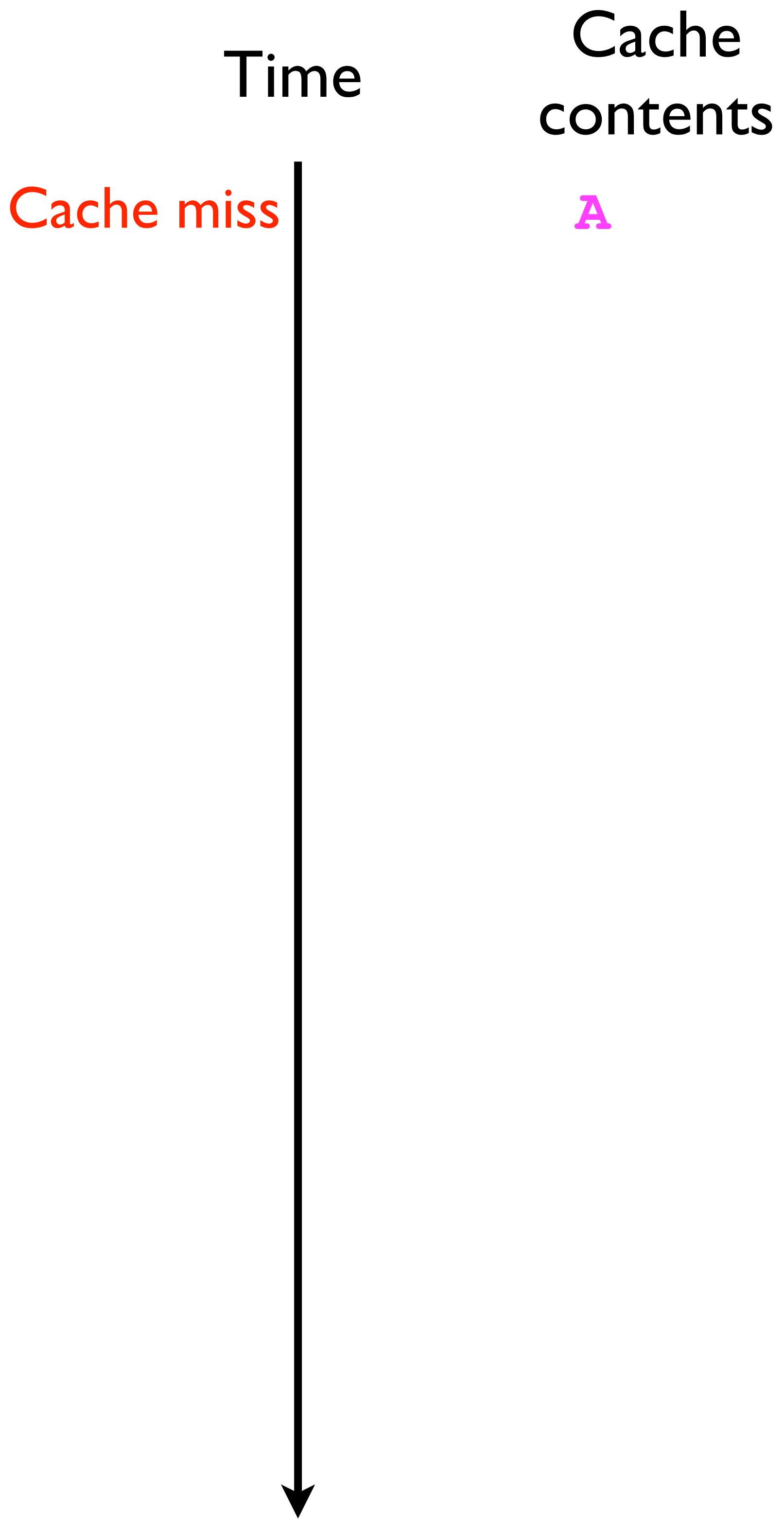
Time

Cache contents

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C

LRU in action

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C



Time

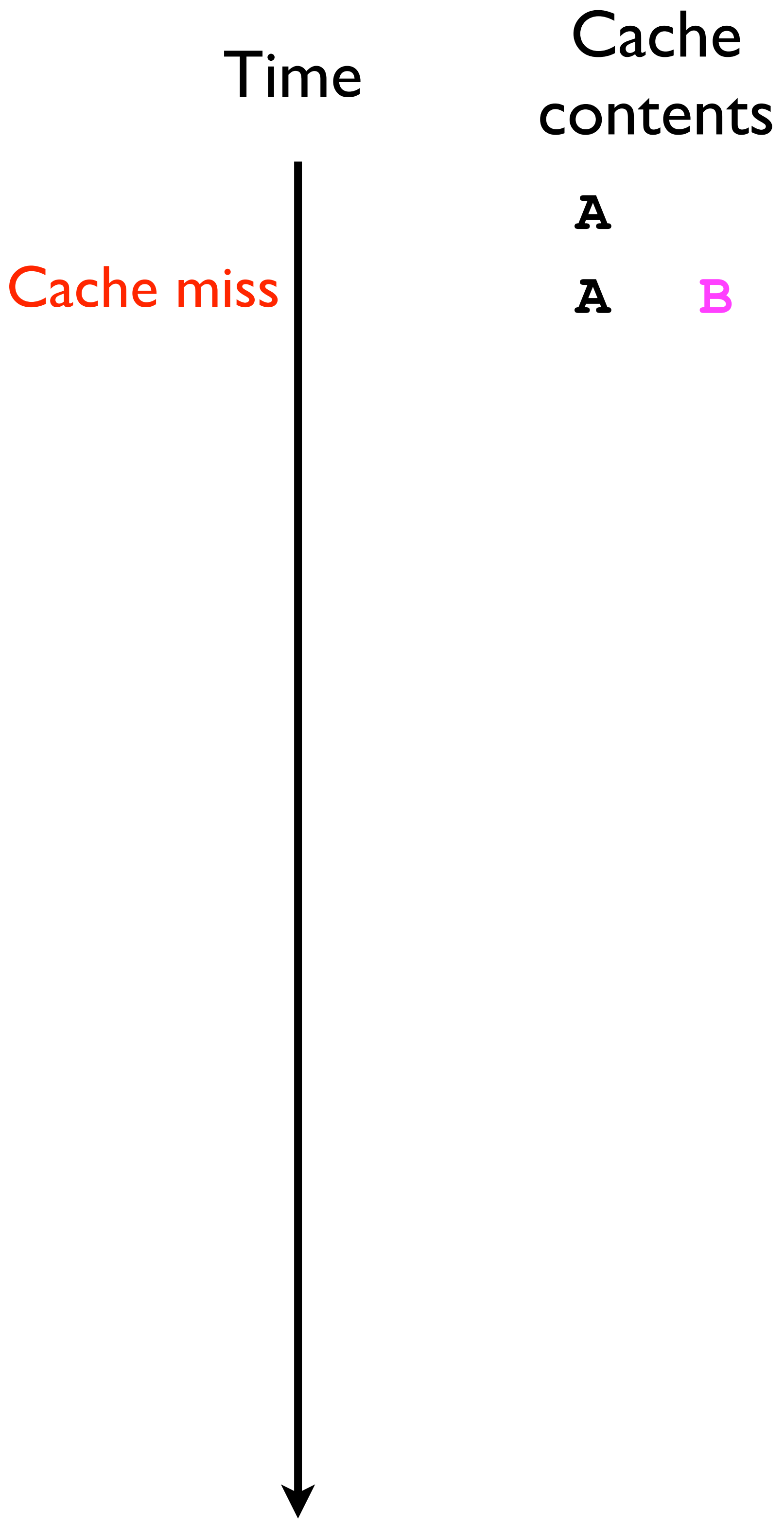
Cache contents

Cache miss

A

LRU in action

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C



Time

Cache contents

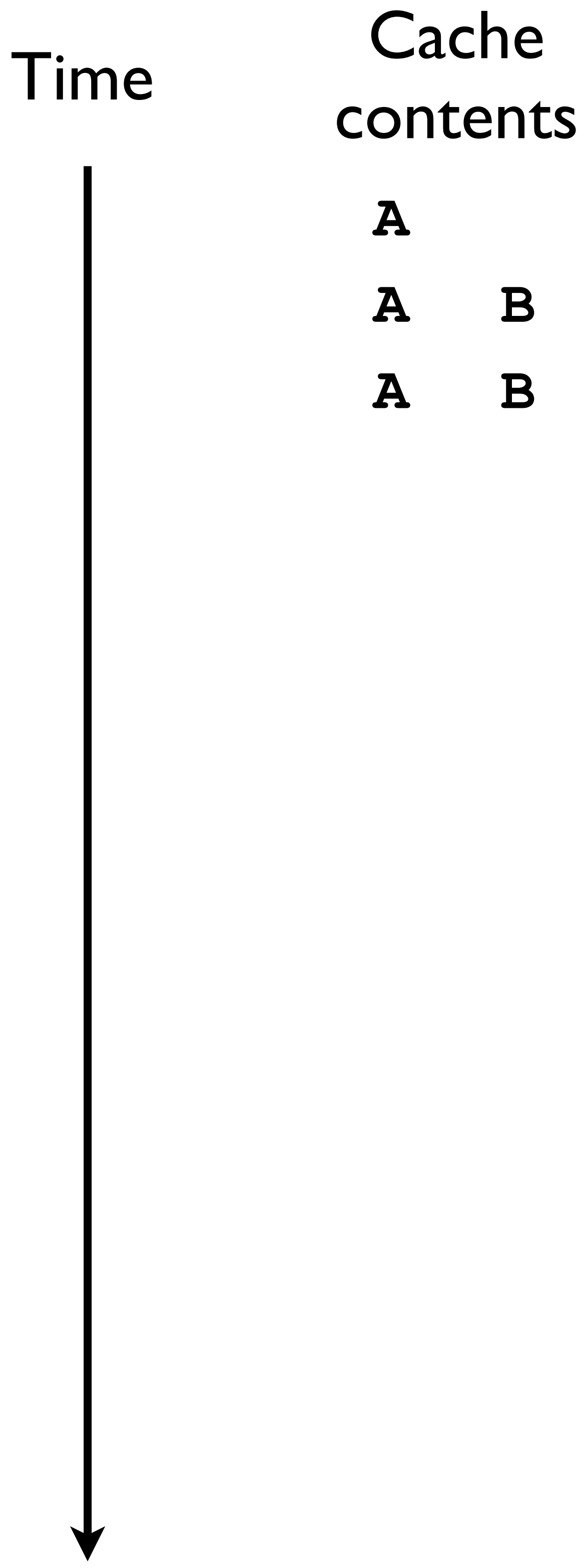
A

Cache miss

A B

LRU in action

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C



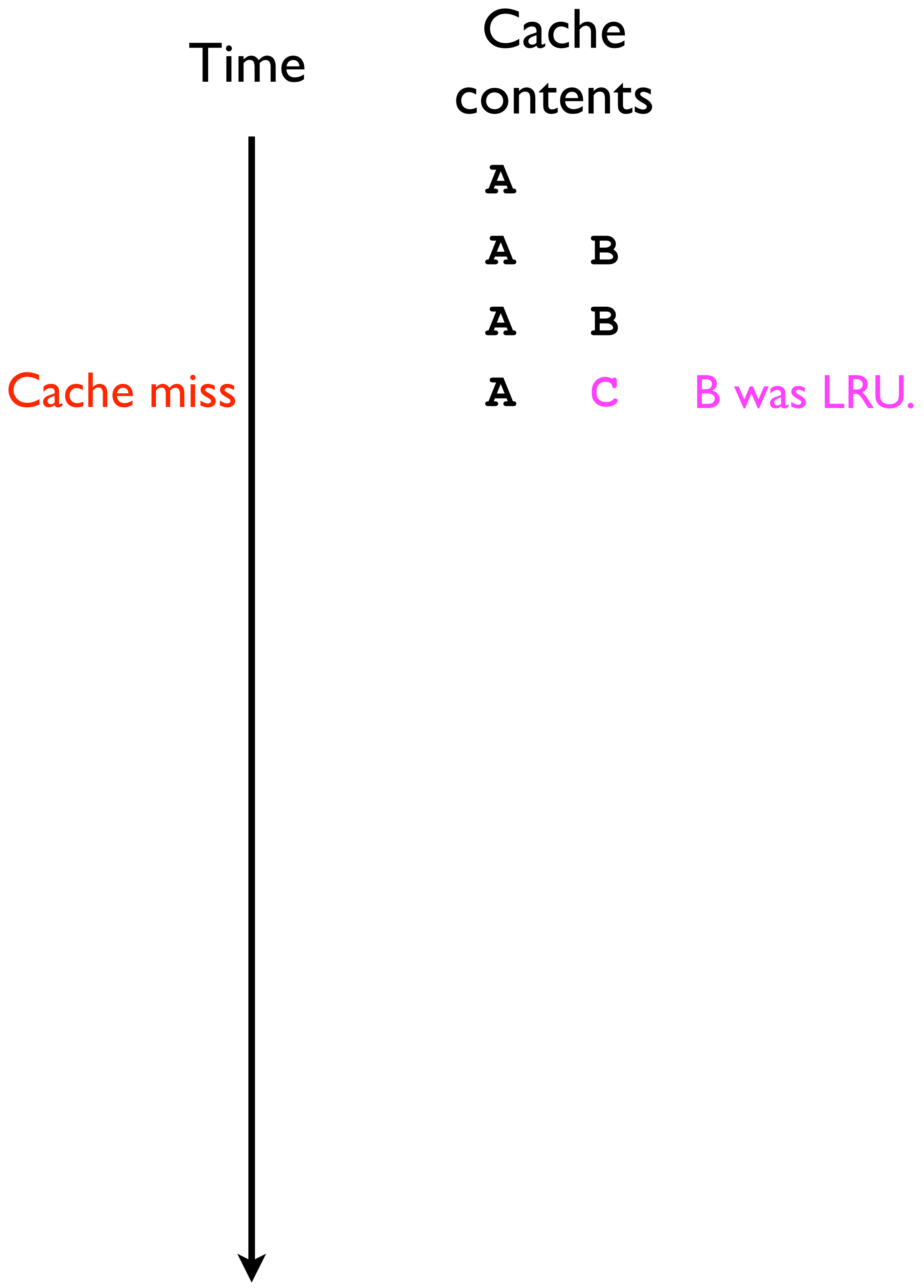
Time

Cache contents

A	
A	B
A	B

LRU in action

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C



	Time	Cache contents		
		A		
		A	B	
		A	B	
Cache miss		A	C	B was LRU.

LRU in action

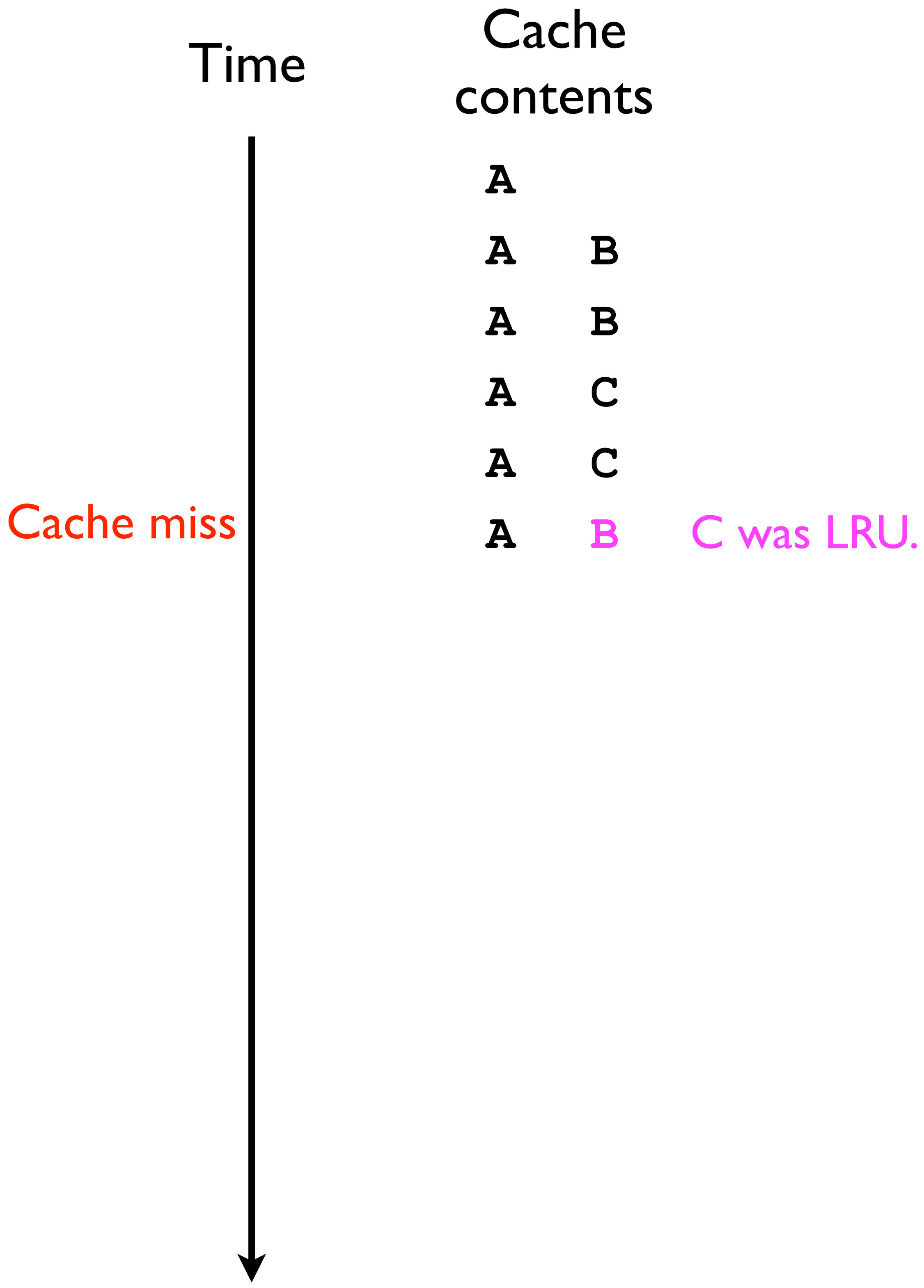
- How would an LRU cache handle the following sequence of requests?
    - A B A C A B B C

Time

Cache contents	
A	
A	B
A	B
A	C
A	C

LRU in action

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C



Time

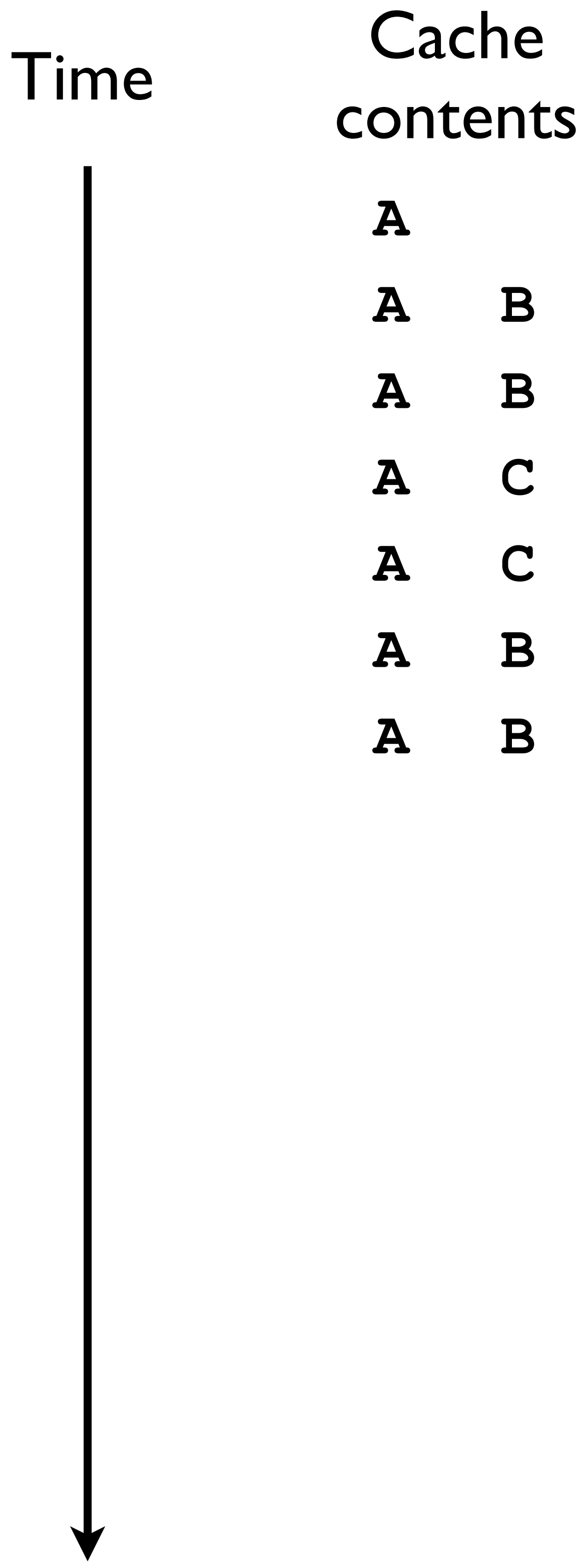
Cache contents

Cache miss

A		
A	B	
A	B	
A	C	
A	C	
A	B	C was LRU.

LRU in action

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C



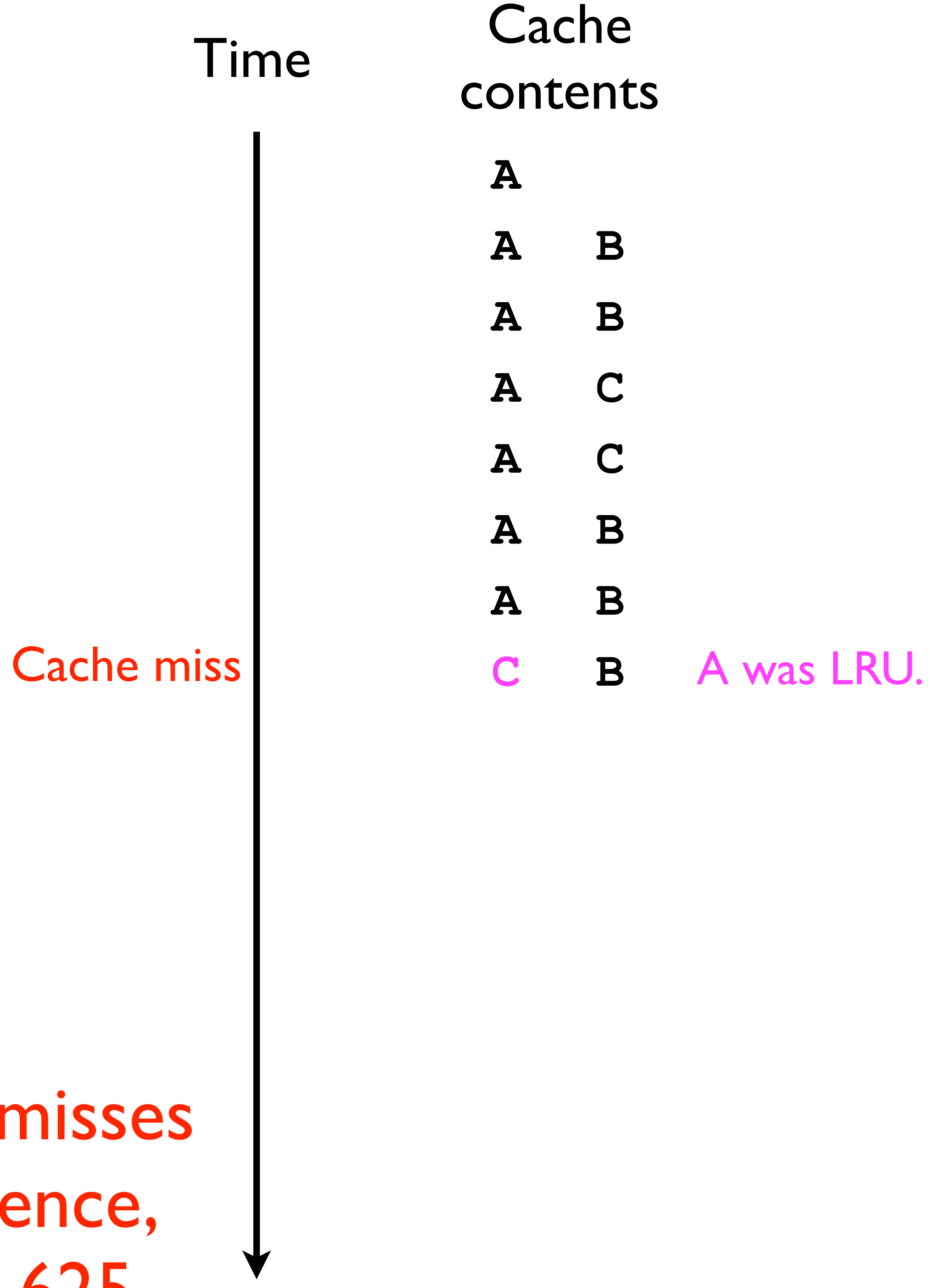
Time

Cache contents

A	
A	B
A	B
A	C
A	C
A	B
A	B

LRU in action

- How would an LRU cache handle the following sequence of requests?
  - A B A C A B B C



Time	Cache contents		
	A		
	A	B	
	A	B	
	A	C	
	A	C	
	A	B	
	A	B	
Cache miss	C	B	A was LRU.

There were 5 cache misses out of 8 accesses; hence, cache miss rate is 0.625.

LRU Cache

- We wish to construct a Cache ADT that uses the LRU eviction policy.
- The cache will mediate access to some other, arbitrary secondary storage container.
- The user will request data by calling `Cache.get(key)` and expect the associated *value* to be returned.
- If `key` is not stored in the cache, then the cache should forward the request to the secondary storage.

LRU Cache interface

- Before designing a Java interface for the LRU cache, let’s first conceptualize how the user might access the secondary storage *without* the cache.
- Suppose the secondary storage has the following interface:

```
interface Storage<K,V> {
  // Fetches and returns the data specified by key
  V get (K key);
}
```

- Here, the *key* might be the URL of a web page we’re fetching, and the *value* might be the web page itself.

LRU Cache interface

- Now, let’s define a Java interface for an LRU cache:

```
// Least-recently-used (LRU) cache.
// The get(key) method should take O(1) time
// for an n-element cache.
//
// Implementing classes should offer a
// constructor with one parameter of type
// Storage that specifies the cache’s
// secondary storage.
interface LRUCache<K,V> {
  V get (K key);
}
```

LRU Cache implementation

- The LRUCache interface imposes the constraint that **`get(key)`** must operate in *O*(1) time for an *n*-element cache.
- Each call to **`get(key)`** must potentially:
  1. Determine whether the desired object (specified by **`key`**) is stored in the cache in *O*(1) time.
  2. If **`key`** *is* in cache, then:
     (a) Make **`key`** the MRU item in *O*(1) time.
     (b) Return the **`key`**'s associated *value* in *O*(1) time.

LRU Cache implementation

3. Else (**`key`** is *not* in cache):

   (a) Call **`value = _secondaryStorage.get(key)`**.

   - This is no problem because it is still *O*(1) regardless of the size of the cache *n*.

   (b) Find the *least*-recently-used (LRU) item in *O*(1) time.

   (c) Replace the LRU item with (**`key,value)`**, which is now the *most*-recently-used (MRU) item in the cache, in *O*(1) time.

LRU Cache implementation

- Hence, an implementation of `LRUCache` might look something like:

```
class LRUCacheImpl<K,V> implements LRUCache<K,V>{
  final Storage<K,V> _secondaryStorage;
  ...

  LRUCacheImpl (Storage<K,V> secondaryStorage) {
    _secondaryStorage = secondaryStorage;
  }

  V get (K key) {
    // If key in cache
    //    Fetch value from cache
    // Else
    //    value = _secondaryStorage.get(key);
    // ...
    // Return value;
  }
}
```

But what will be the “underlying storage” for the cache entries themselves?

LRU Cache implementation

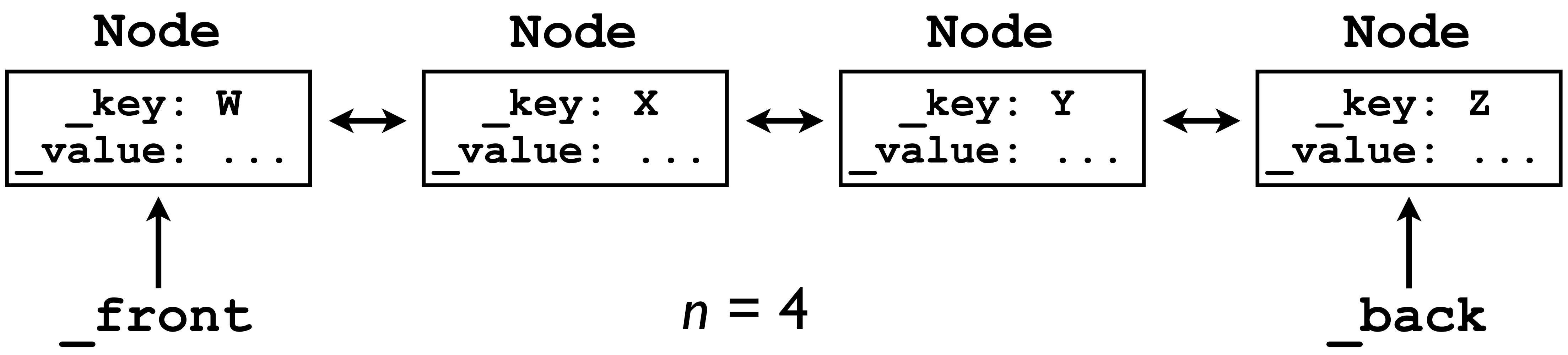
- Our “underlying storage” will consist of 2 components:

1. A *queue* of **`Nodes`** to hold the *relative order* in which data are accessed.

    - For *n*-element cache, max length of queue is *n*.
    - LRU at the *front*, MRU at the *back* of the queue.
    - Each **`Node`** will contain both a *key* (e.g., URL) and corresponding *value* (e.g., webpage).

`W` is LRU item.

`Z` is MRU item.



Node

_key: W
_value: ...

Node

_key: X
_value: ...

Node

_key: Y
_value: ...

Node

_key: Z
_value: ...

_front

$n = 4$

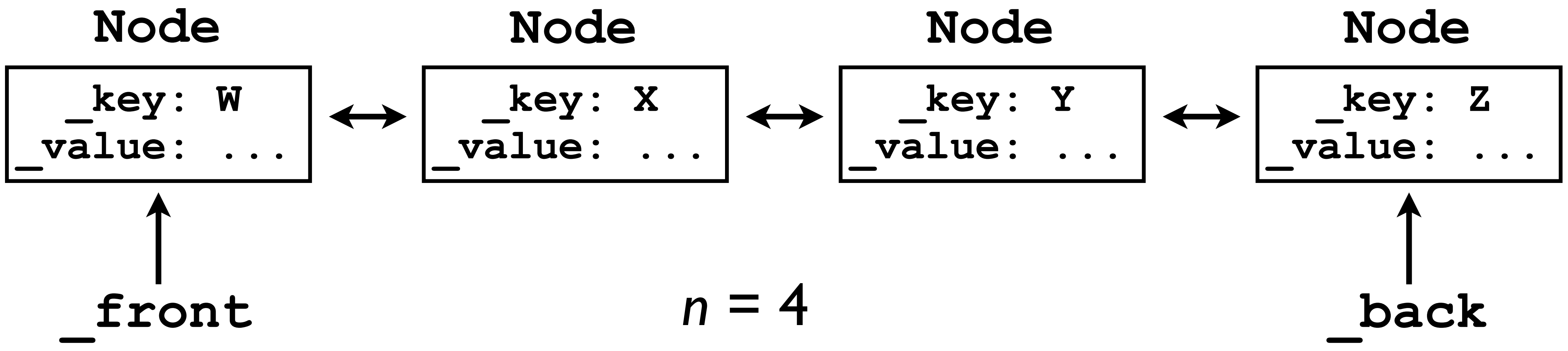
_back

LRU Cache implementation

- All the important cache data is stored in the queue.
- Whenever data X is requested, we move its **`Node`** to the *back* of the queue because it's now the MRU item.
- Whenever data V (not in the cache) is requested, we fetch it from secondary storage, and then store it in the cache.
    - We must evict the LRU item to make room.

`W` is LRU item.

`Z` is MRU item.



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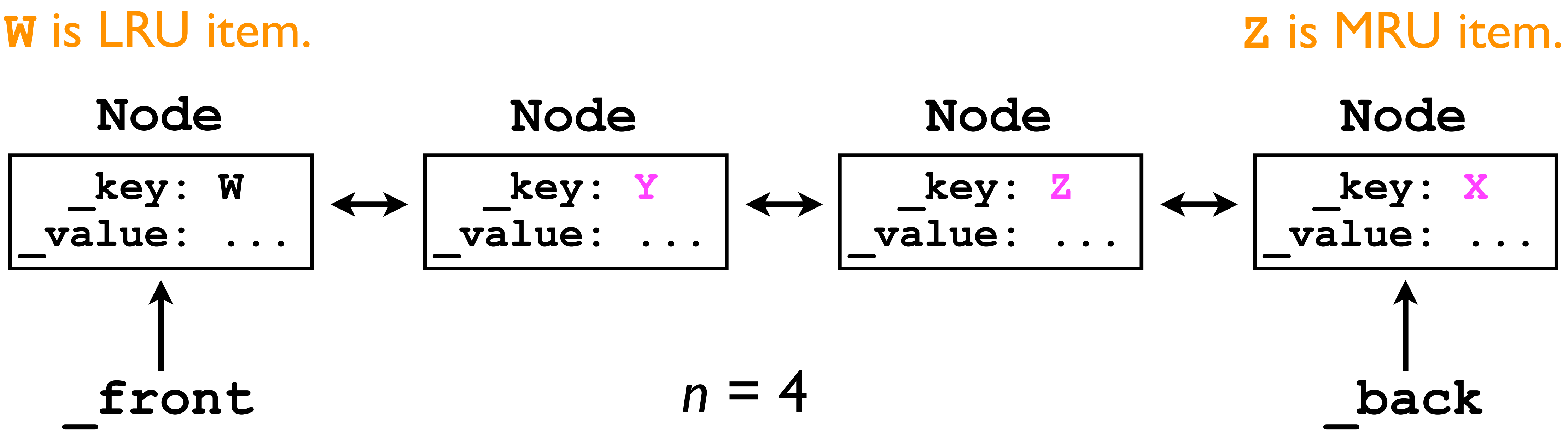

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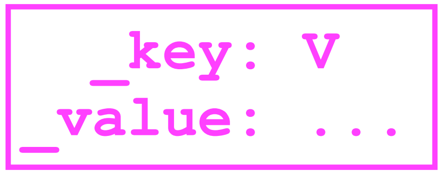

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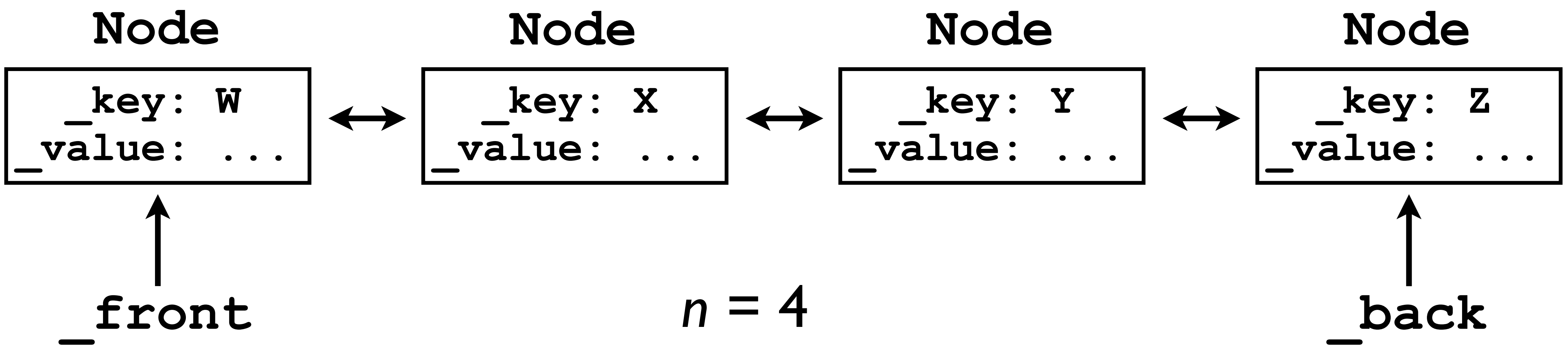

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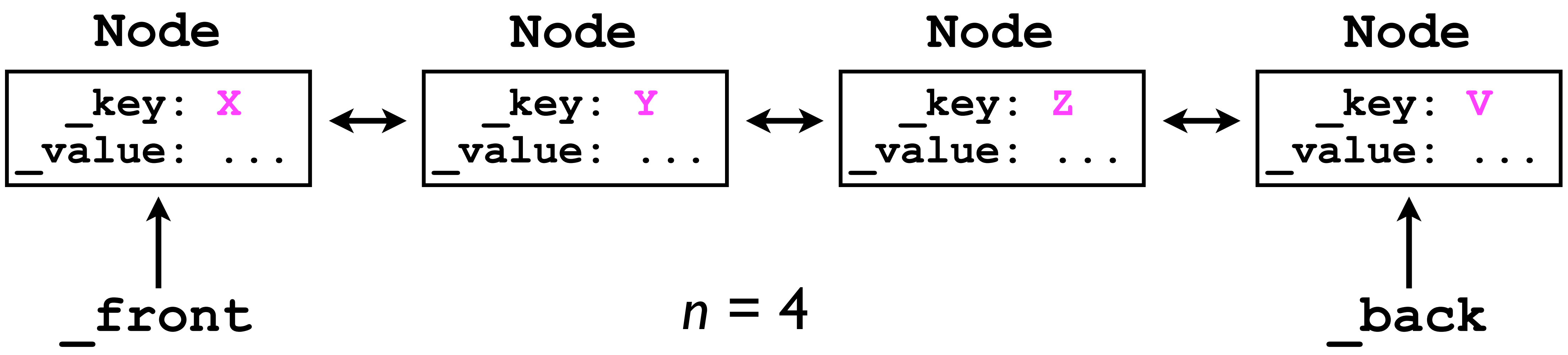
  - We must evict the LRU item to make room.

`W` was LRU item and was evicted.

`V` is now MRU item.



`Node` **`_key: X`** **`_value: ...`** ↔ **`Node`** **`_key: Y`** **`_value: ...`** ↔ **`Node`** **`_key: Z`** **`_value: ...`** ↔ **`Node`** **`_key: V`** **`_value: ...`**

`_front`

$n = 4$

`_back`

Reality check

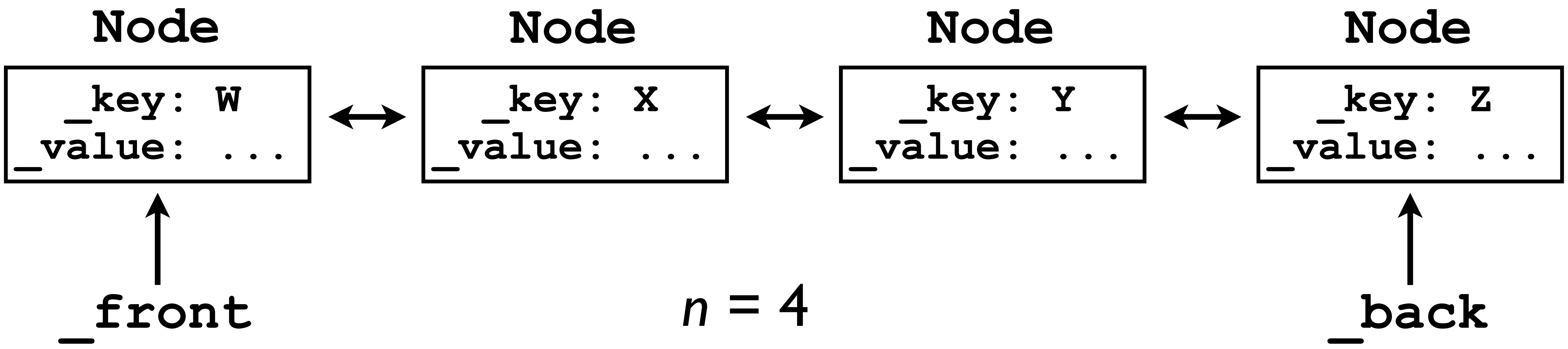
- Suppose the cache stores *n* = 3 elements, and suppose the user requests the following webpages in the following order:

```
cnn.com
google.com
gmail.com
yahoo.com
npr.org
wikipedia.org
cnn.com
gmail.com
npr.org
cnn.com
imdb.com
```

- Show the queue at each step.

LRU Cache implementation

- Unfortunately, a queue by itself will not suffice to implement the **`LRUCache`** interface.
  - When we want to update a **`Node`**‘s position in the queue to MRU, we have to *find* the node (*O*(*n*)).
- However, we can use an additional **`HashTable<K,Node>`** to “jump” to the desired **`Node`** in *O*(1) time.



`Node` **`Node`** **`Node`** **`Node`**

`_key: W` `_value: ...` ↔ `_key: X` `_value: ...` ↔ `_key: Y` `_value: ...` ↔ `_key: Z` `_value: ...`

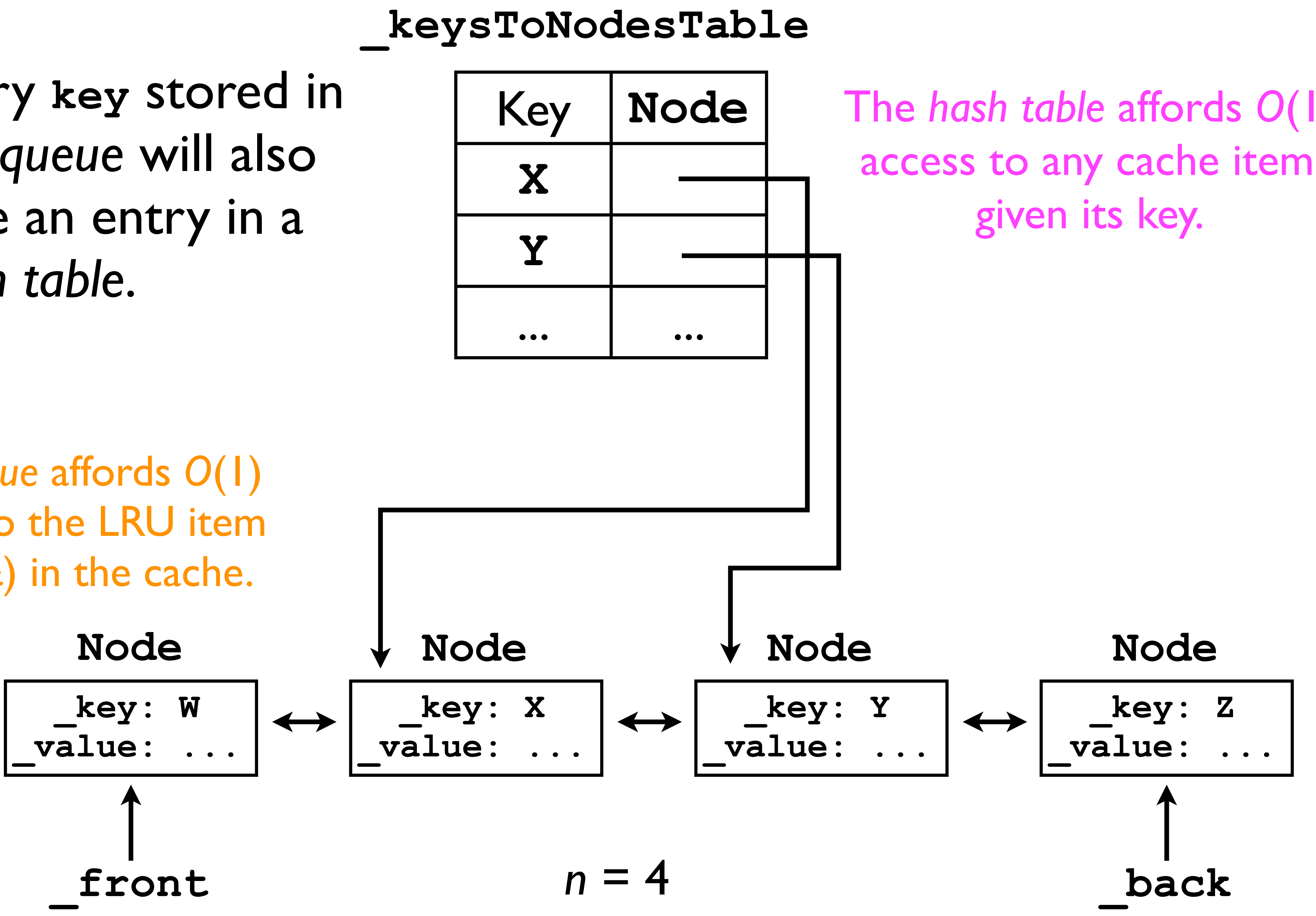
`_front`

n = 4

`_back`

LRU Cache implementation

- Every **`key`** stored in the *queue* will also have an entry in a *hash table*.



`_keysToNodesTable`

Key	**Node**
X	
Y	
...	...

The *hash table* affords *O*(1) access to any cache item, given its key.

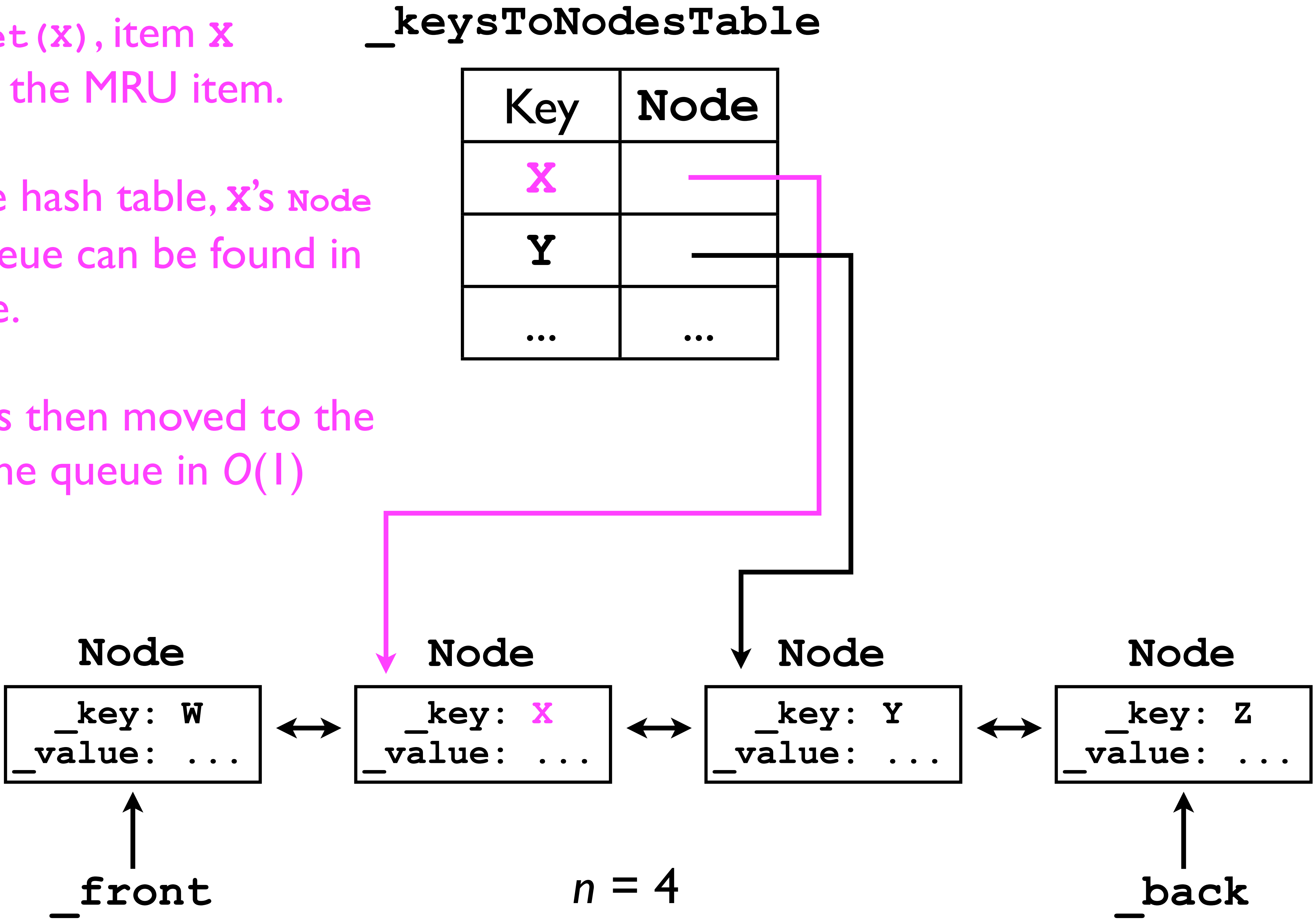
The *queue* affords *O*(1) access to the LRU item (**`_front`**) in the cache.

LRU Cache implementation

Whenever the user calls `cache.get(X)`, item `X` becomes the MRU item.

Using the hash table, `X`'s `Node` in the queue can be found in *O*(1) time.

Its `Node` is then moved to the *back* of the queue in *O*(1) time.



`_keysToNodesTable`

Key	`Node`
`X`	
`Y`	
...	...

LRU Cache implementation

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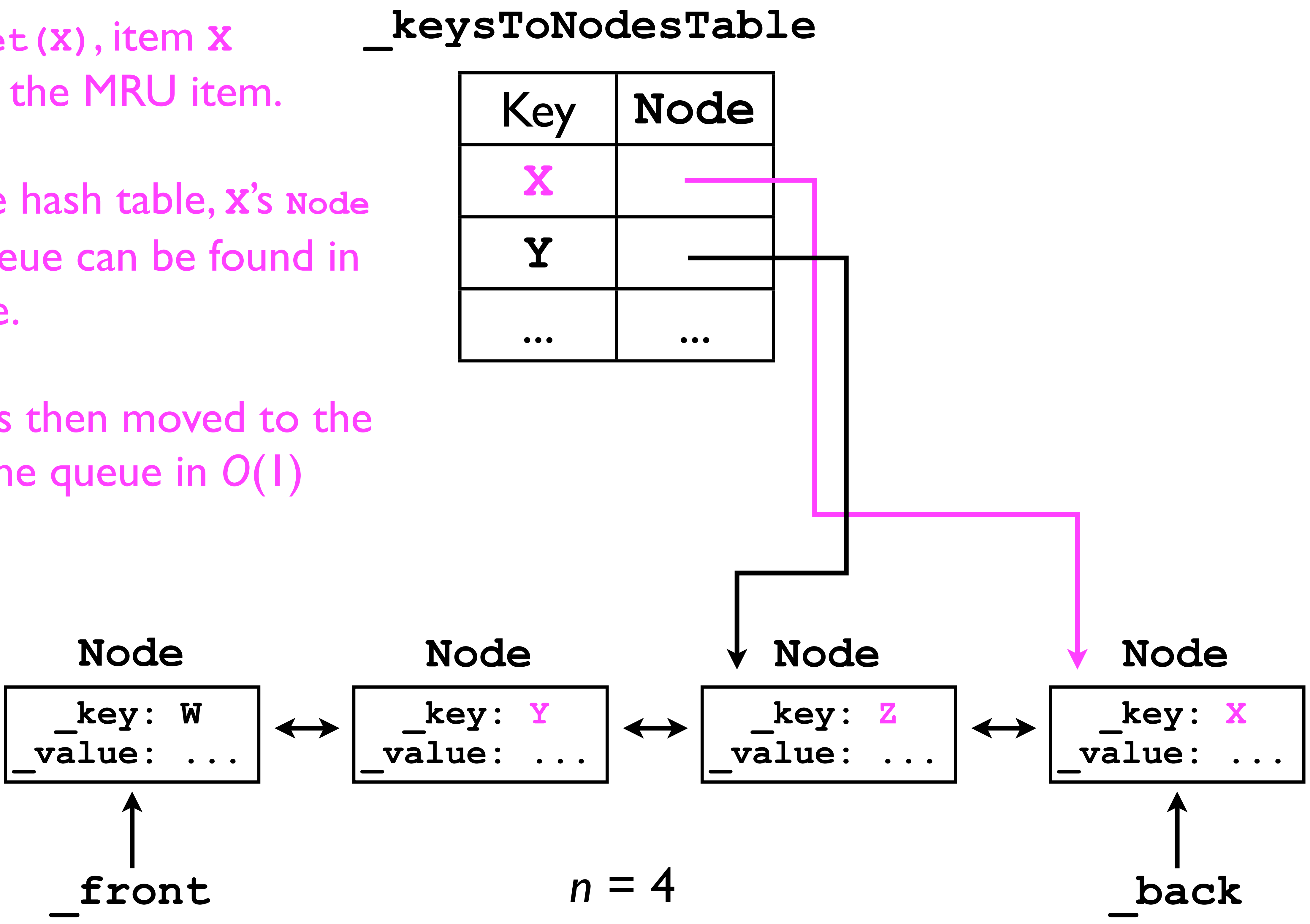
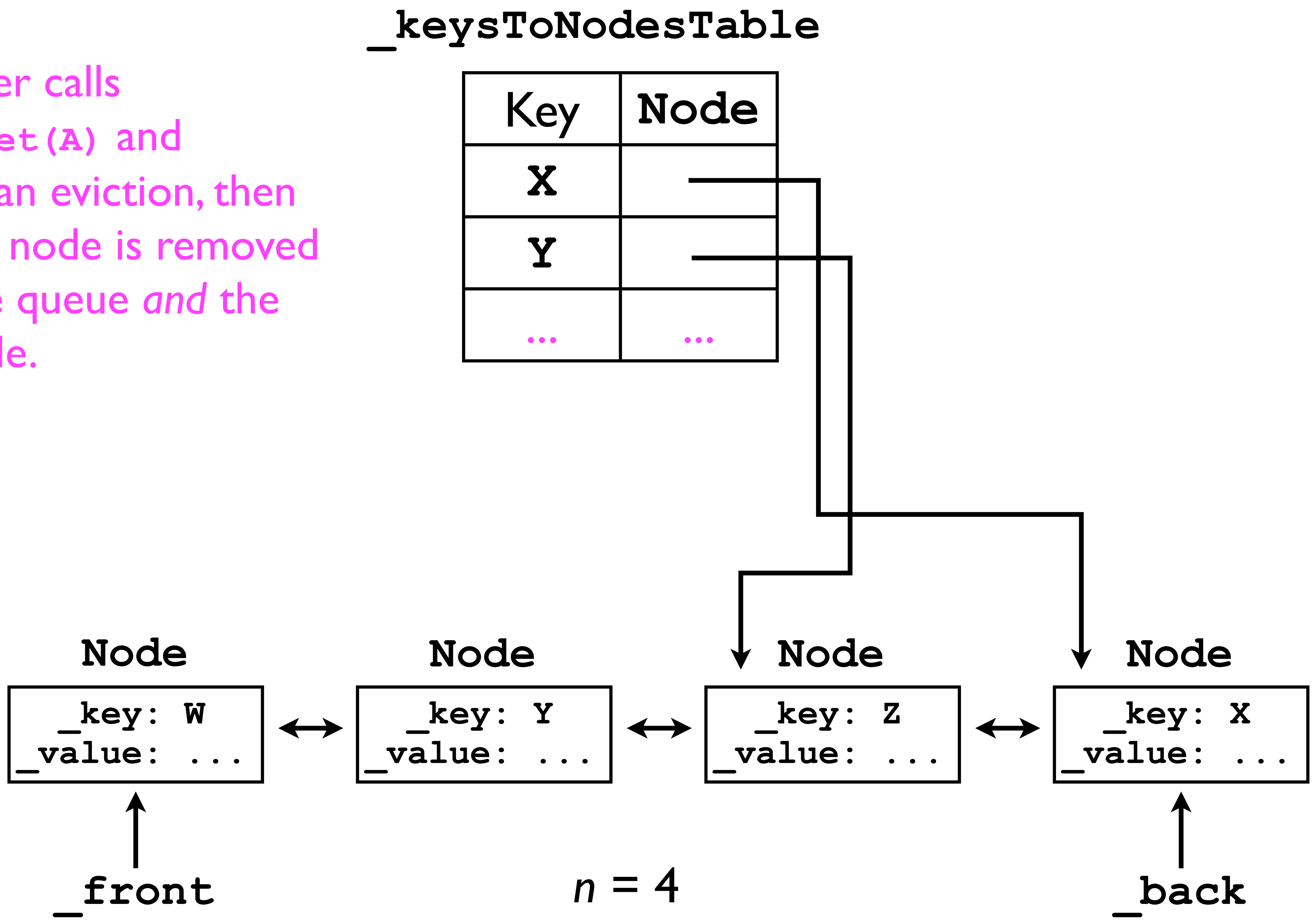


`_keysToNodesTable`

Key	Node
X	
Y	
…	…

LRU Cache implementation

If the user calls `cache.get(A)` and triggers an eviction, then the LRU node is removed from the queue *and* the hash table.



`_keysToNodesTable`

Key	Node
X	
Y	
...	...

$n = 4$

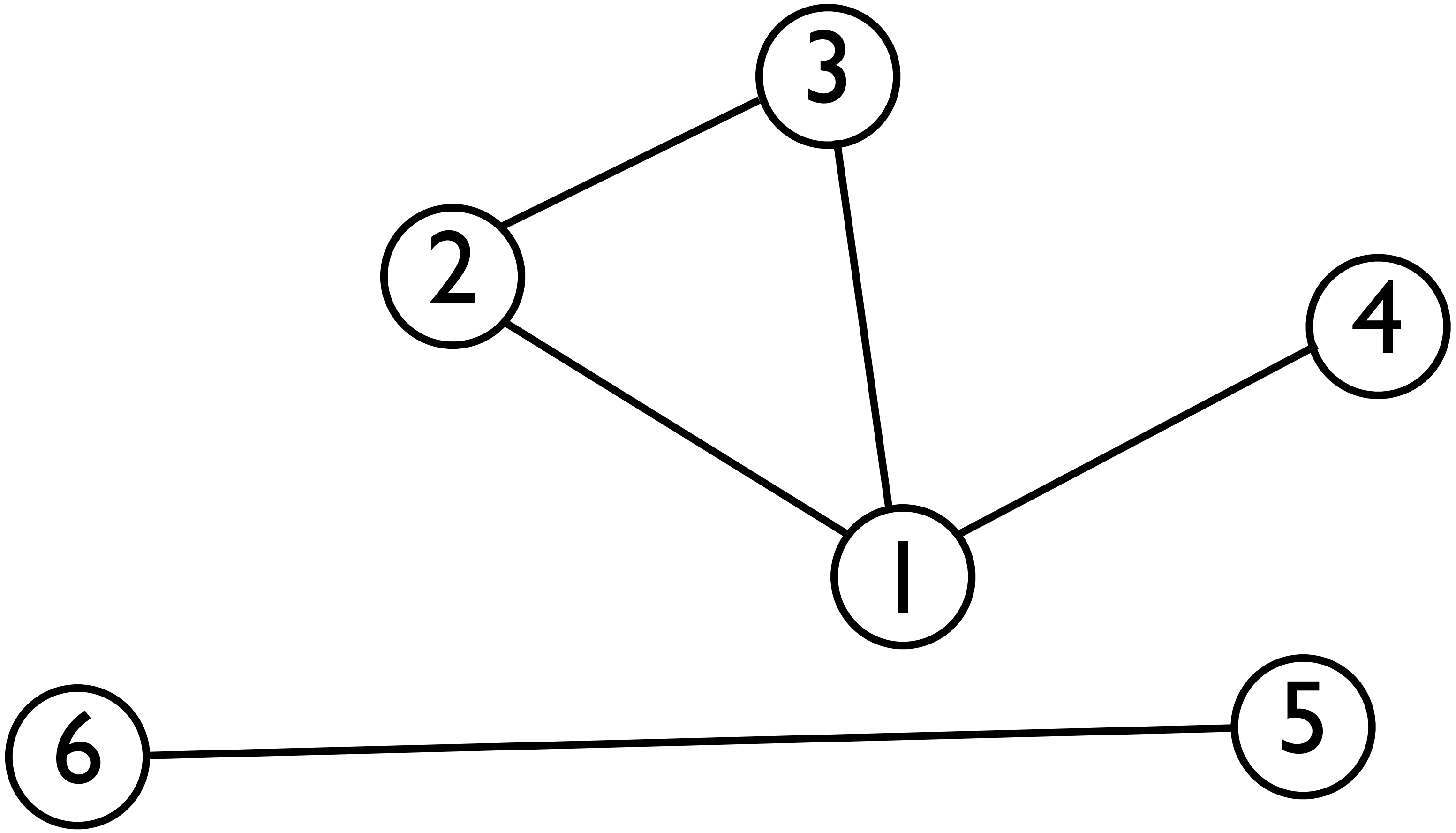
LRU Cache implementation

- In summary:
  - An LRU cache is an example of combining data structures to harness their individual strengths.
  - To implement an LRU cache with *O*(1) time for **`V get (K key)`**, we need fast access both to the LRU item, *and* to an *arbitrary* item specified by **`key`**.
  - A *queue* gives us *O*(1) access to the LRU item (front of queue).
  - A *hash table* gives us *O*(1) access to an arbitrary **`Node`** in the queue.

Graphs.

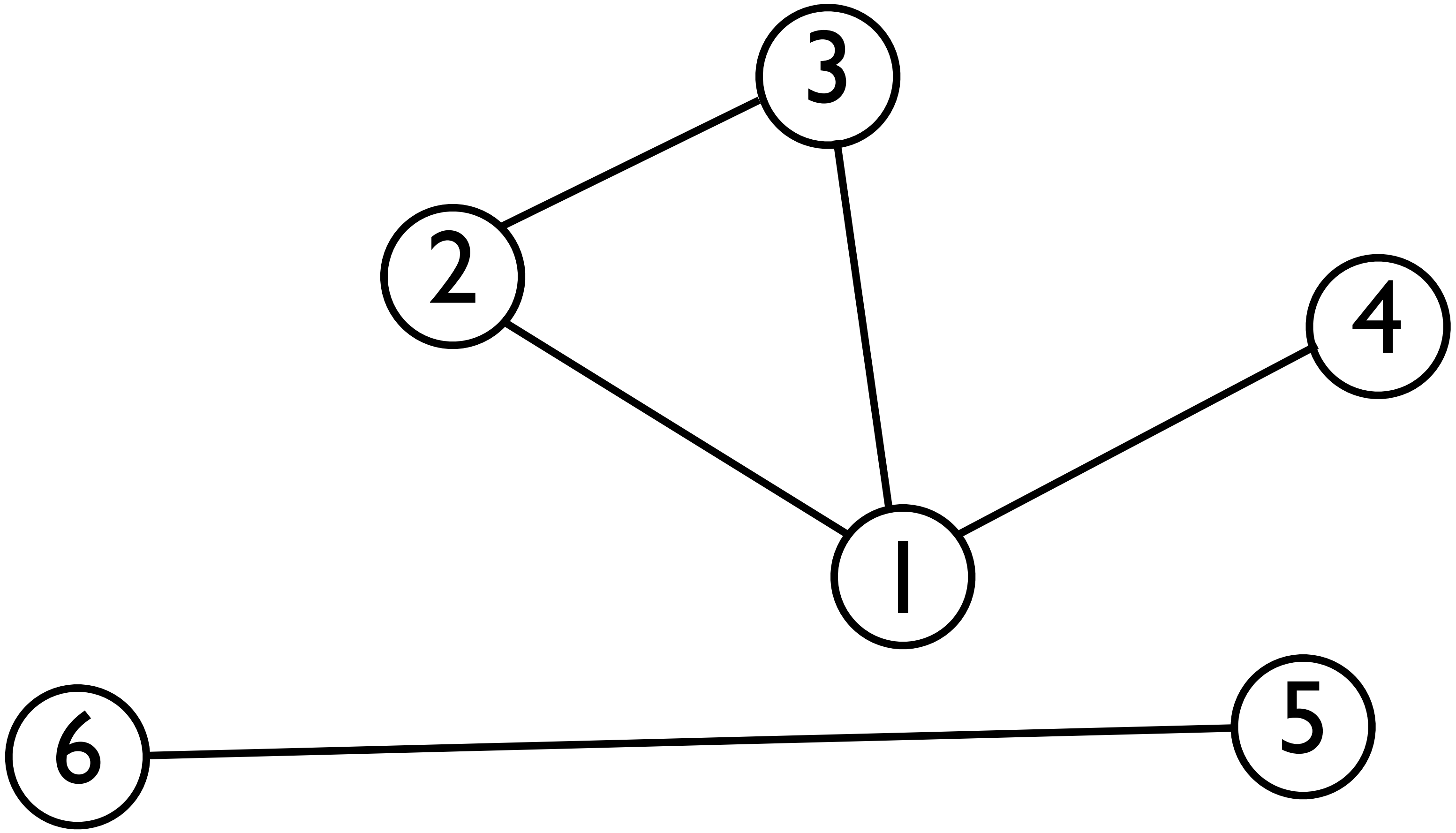
Graphs

- The last fundamental data structure we will cover in this course is a *graph*.
- Mathematically, a **graph** consists of a set *N* of **nodes** (aka **vertices**) connected by a set *E* of **edges**.



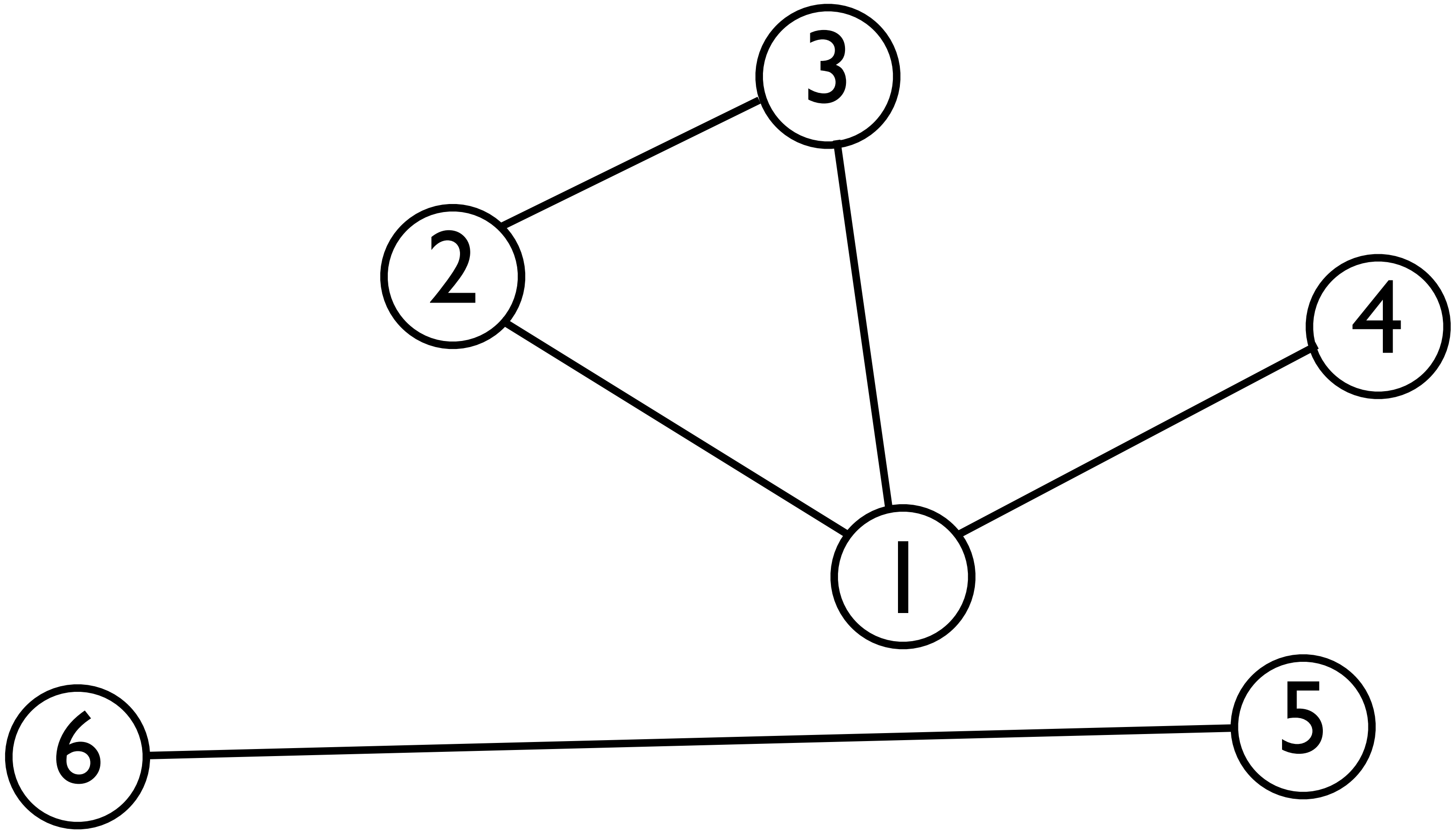
Graphs

- In computer science, graphs are useful for describing *relationships* (edges) among *things* (nodes).
  - E.g., each node might represent a *Facebook user*, and each edge might represent whether two Facebook users are *friends*.



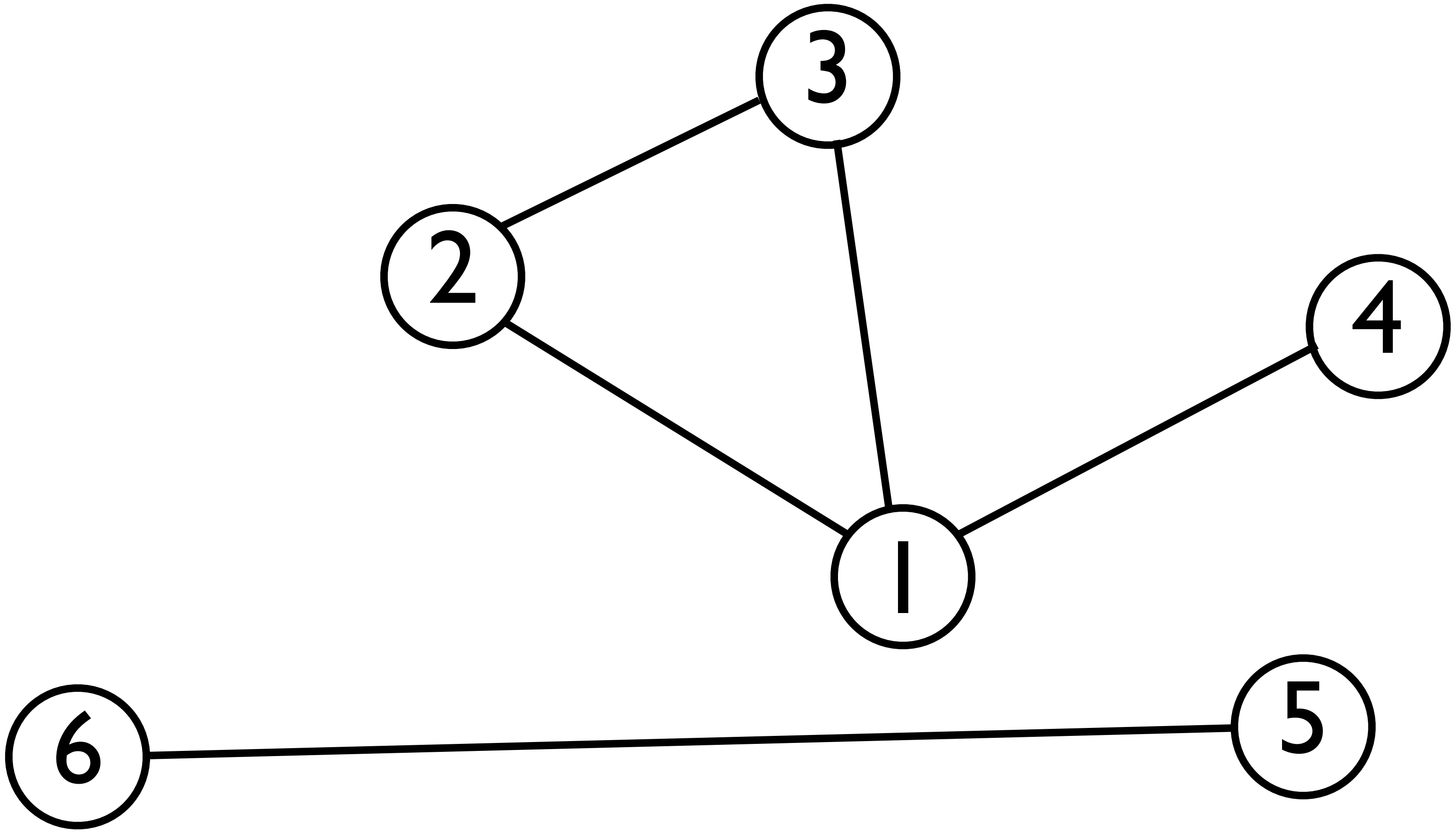
Graphs

- E.g., each node might represent a *computer server*, and each edge represents whether two nodes are *linked* by Ethernet.



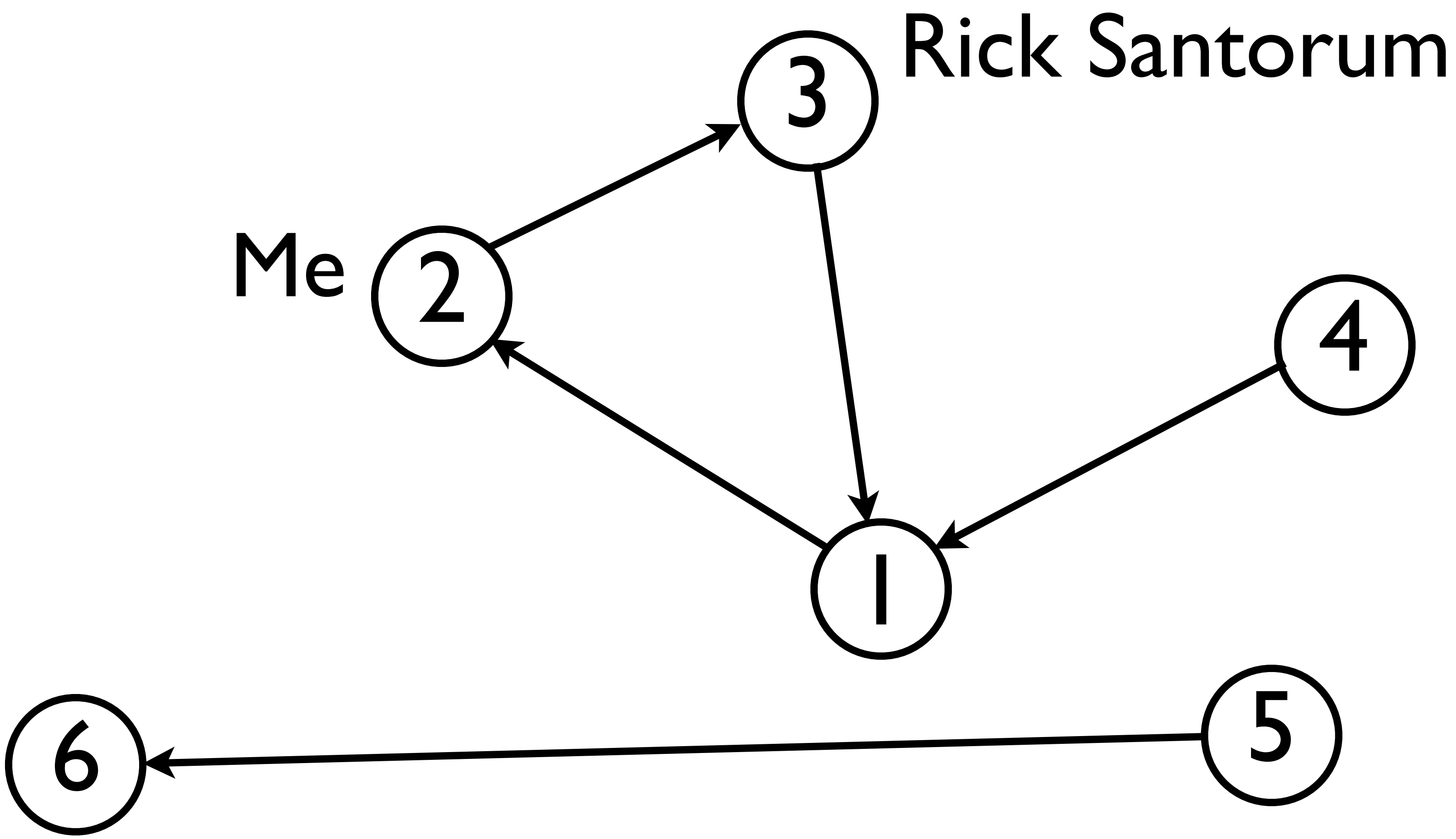
Graphs

- Like *trees*, graphs consist of *nodes* and *edges*.
- Unlike trees, graph can contain *cycles*.
- Graphs can be either **undirected** (as below)...



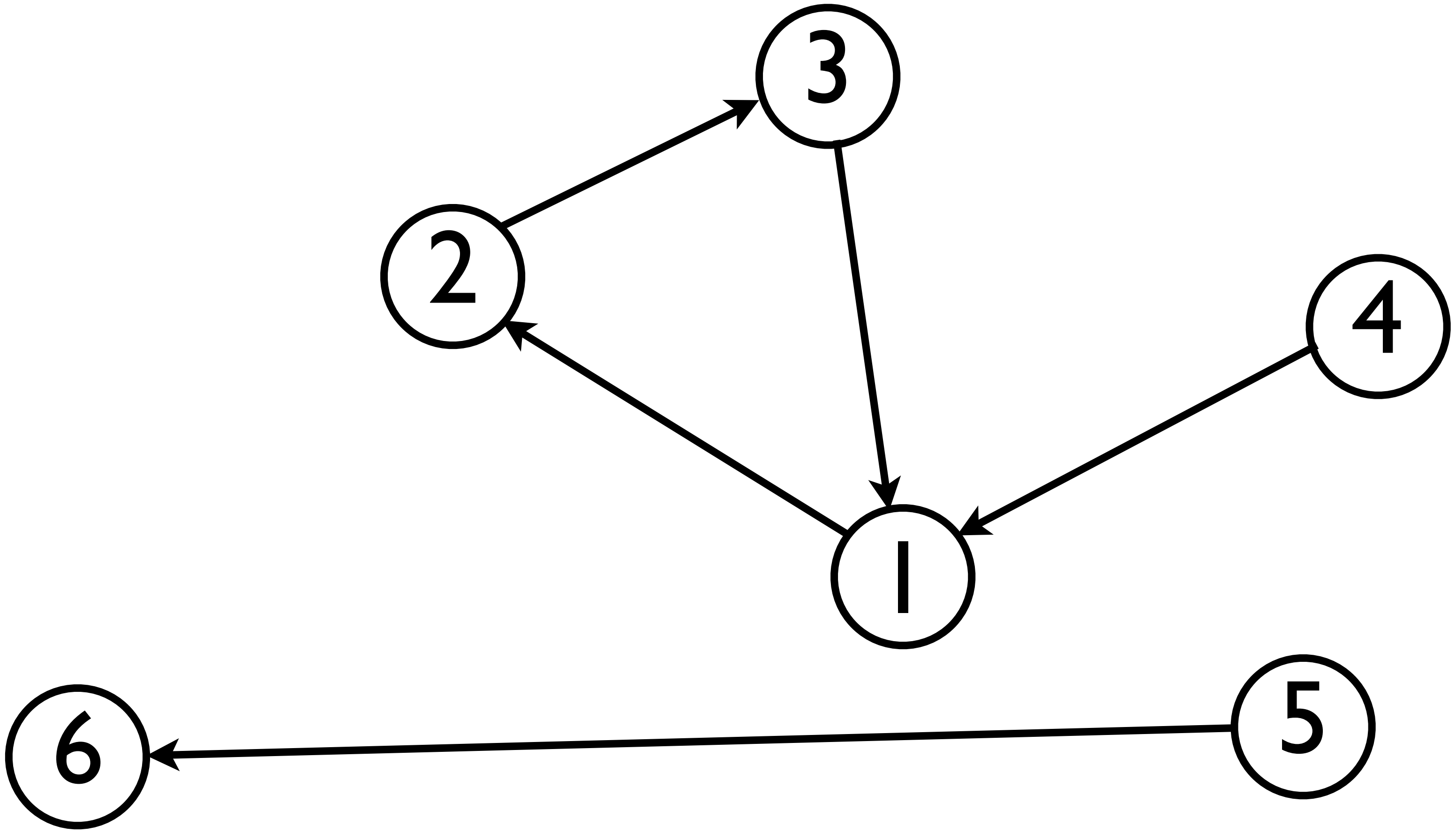
Graphs

- ...or **directed** (as below).
- *Directed graphs* are useful for describing *asymmetric* relationships, e.g., "I know who Rick Santorum is, but he doesn't know who I am."



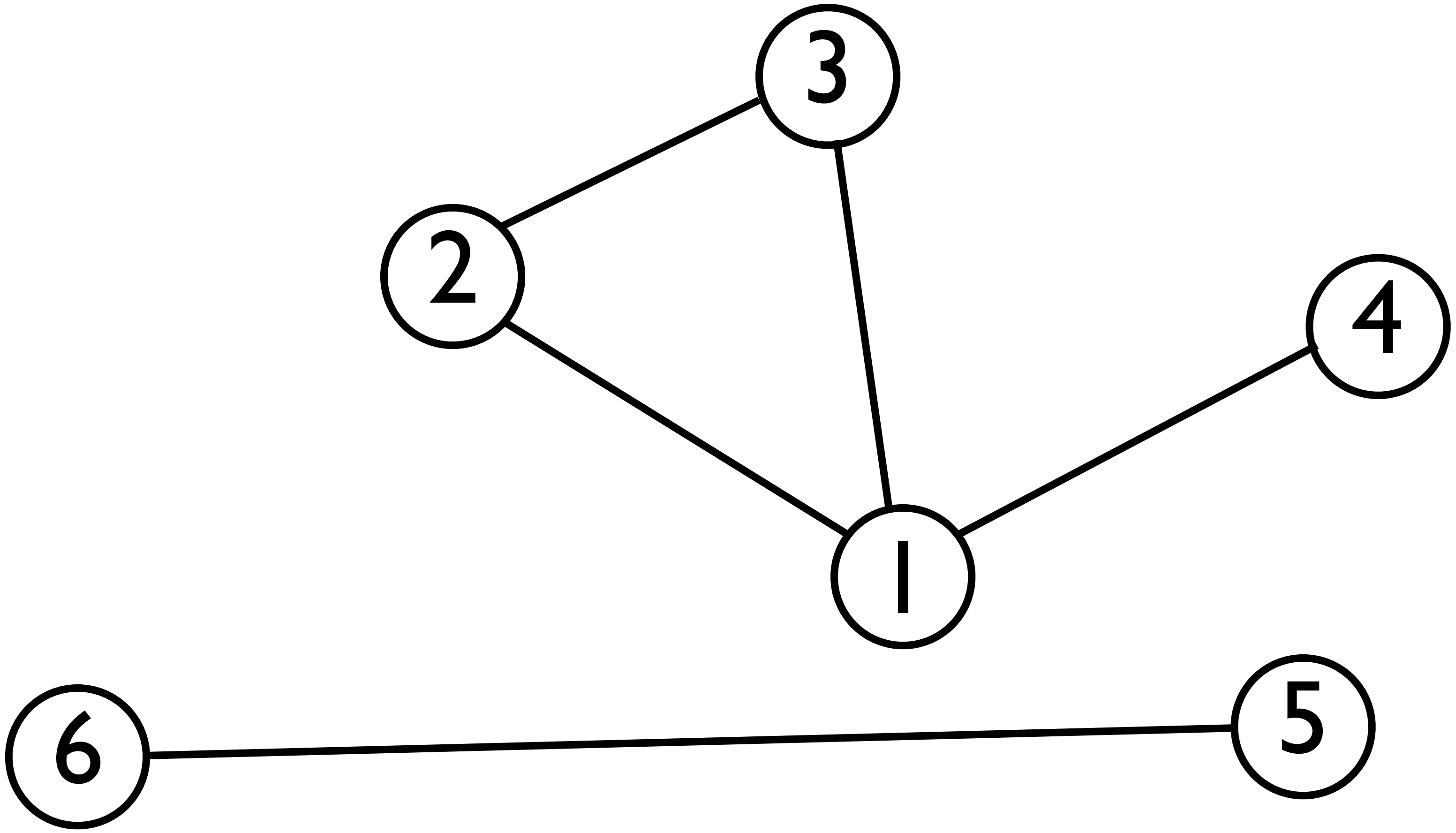
Graphs

- In the graph below, *N* = { 1, 2, 3, 4, 5, 6 }.
- An *edge* in a directed graph from node *m* to node *n* can be described as an *ordered pair* (*m*, *n*).
- In the graph below, *E* = { (2, 3), (3, 1), (1, 2), (4, 1), (5, 6) }.



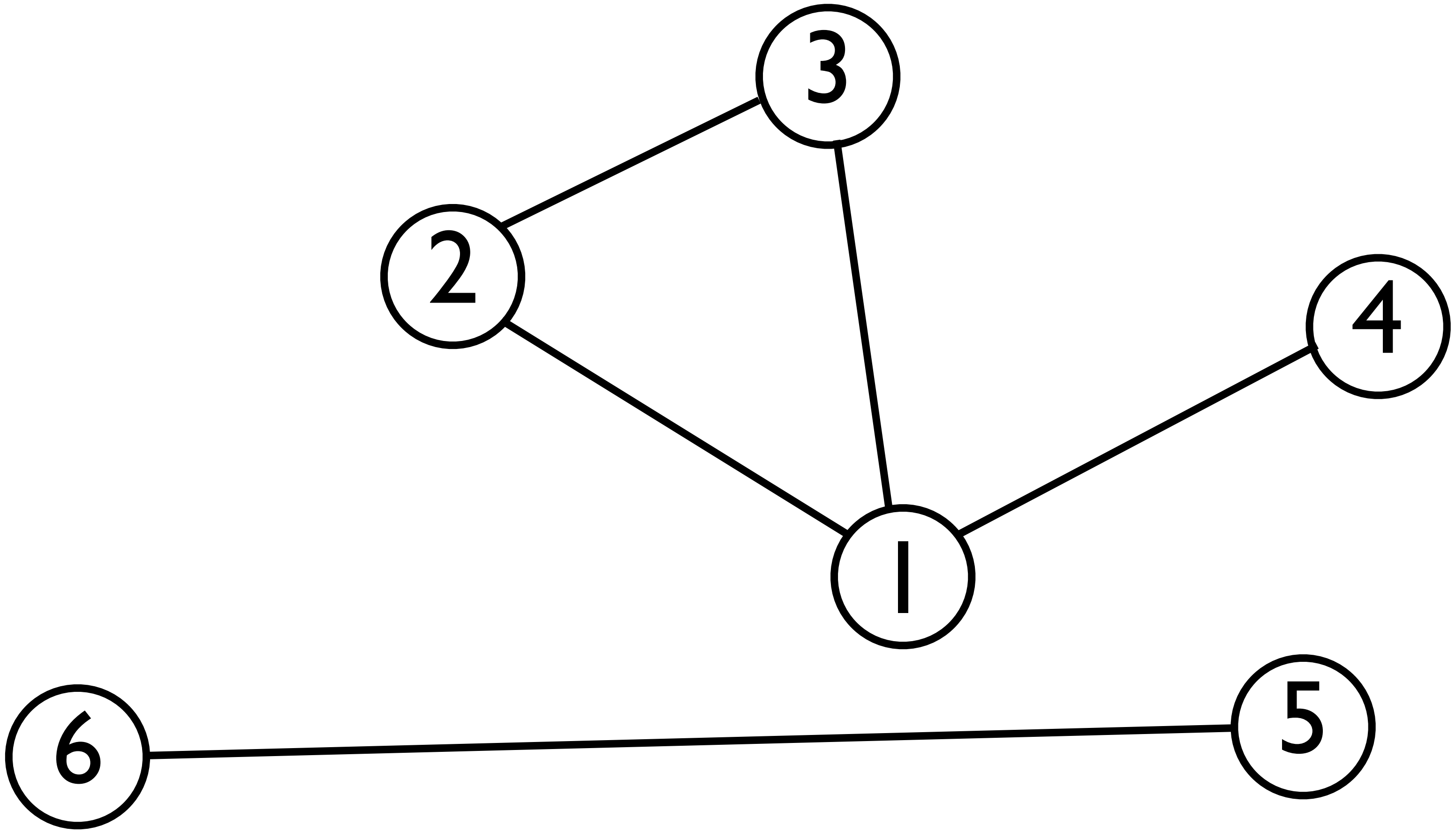
Graphs

- If a graph is undirected, then for every edge (*m*, *n*) ∈ *E*, we also have (*n*, *m*) ∈ *E*.
- For the graph below, *E* = { (2, 3), (3, 2), (1, 3), (3, 1), (1, 2), (2, 1), (1, 4), (4, 1), (5, 6), (6, 5) }.



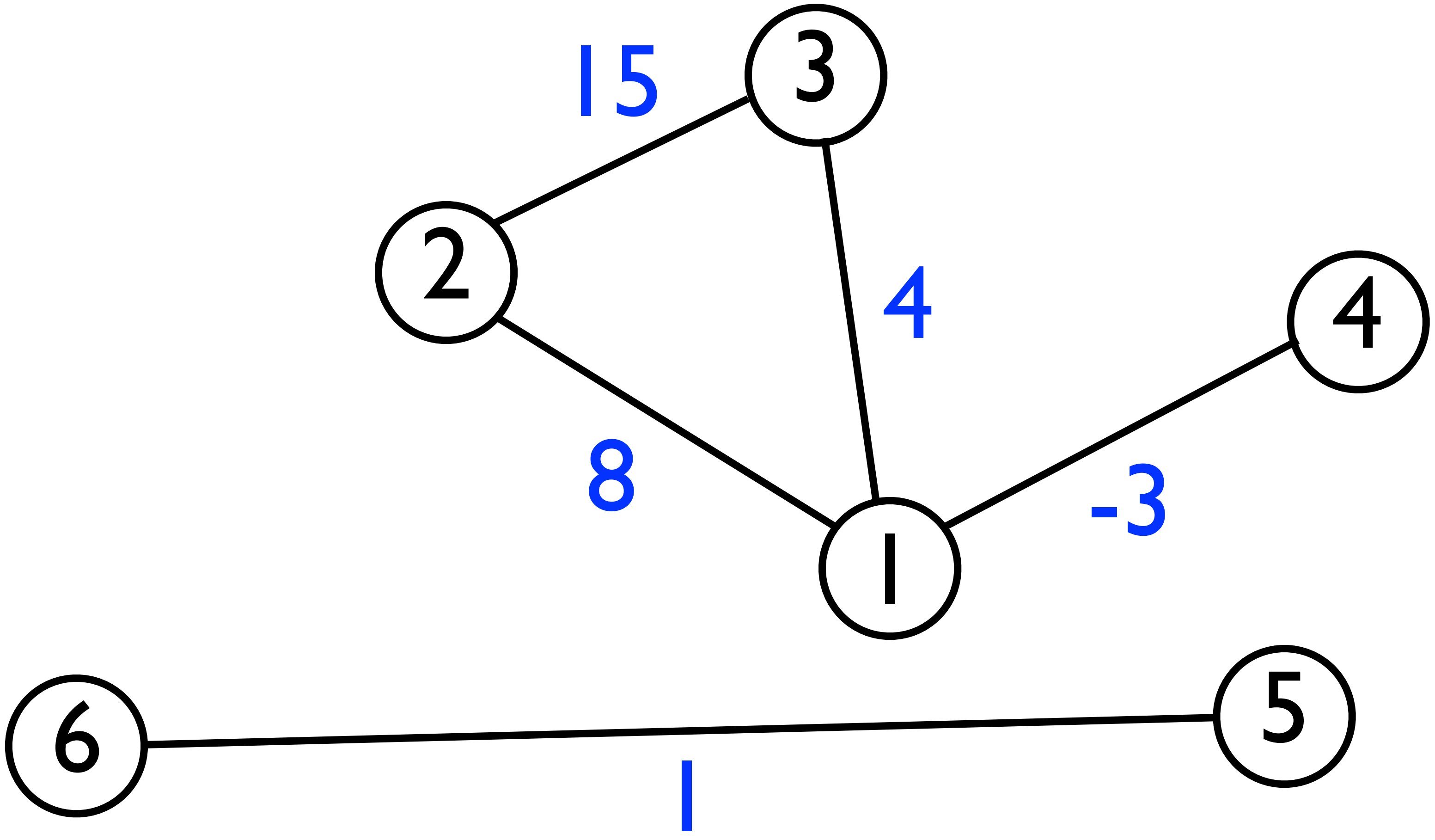
Graphs

- Whenever $(m, n) \in E$, we say that node m is **adjacent** (or **connected**) to node n.



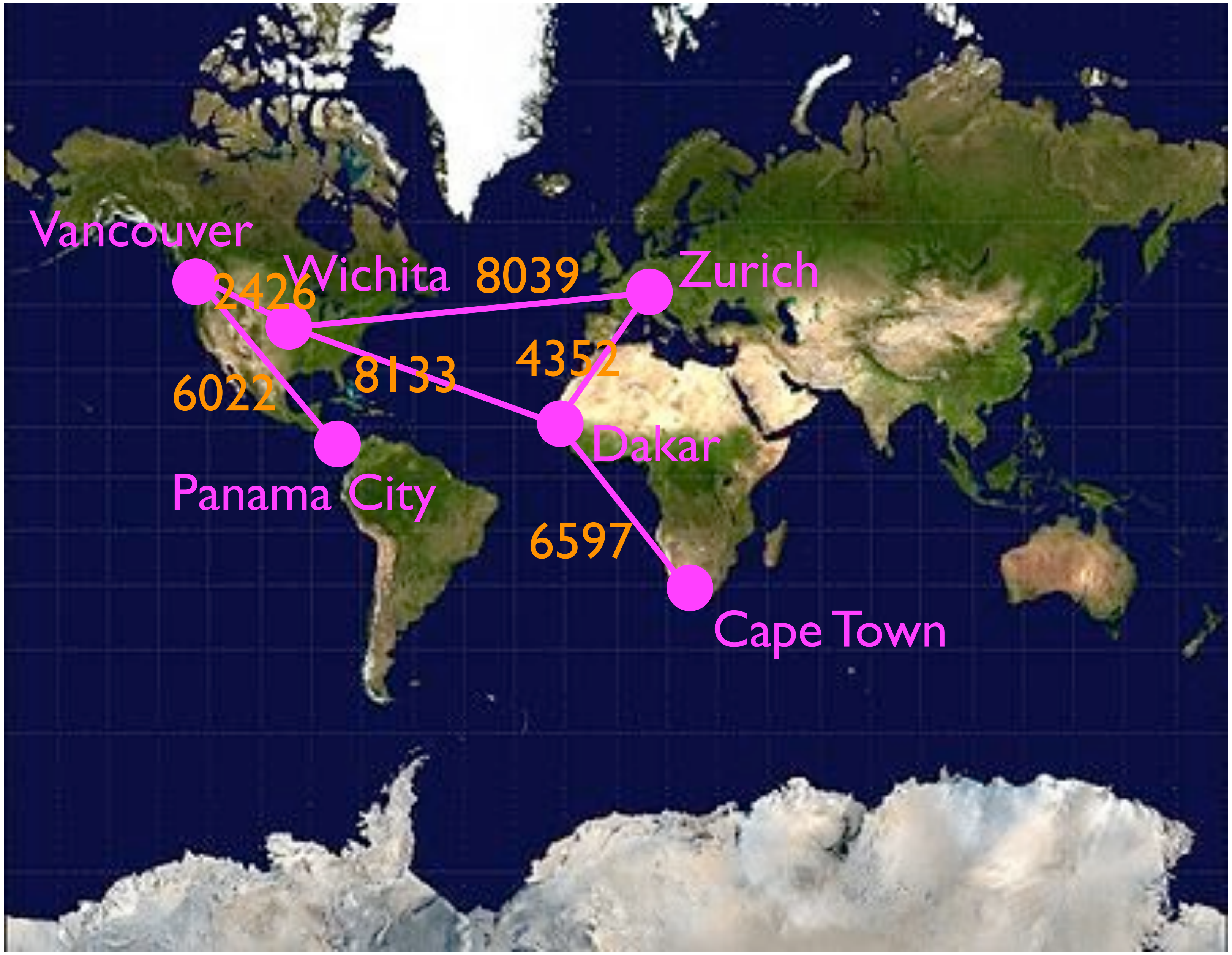
Graphs

- In some graphs, edges have **weights** associated with them to represent distance, cost, etc.
  - In this case, an edge can be represented as an ordered triplet (m, n, w_{mn}) where w_{mn} is the weight from m to n.



Graphs

- An example of a weighted graph is an airline map that shows *cities* connected by *flights*, and the weight of each edge is the *distance* (km) between those cities.



Representing graphs

- To use graphs as a data structure, we must devise a way of representing a graph in memory.
- Let *N* be the set of nodes and *E* be the set of edges.
- The number of nodes is |*N*|, and the number of edges is |*E*|.
- To represent the set of *nodes* in memory, we can use an |*N*|-element array, where each node is assigned a unique index.
  - This is both time- and space-efficient.

Representing graphs

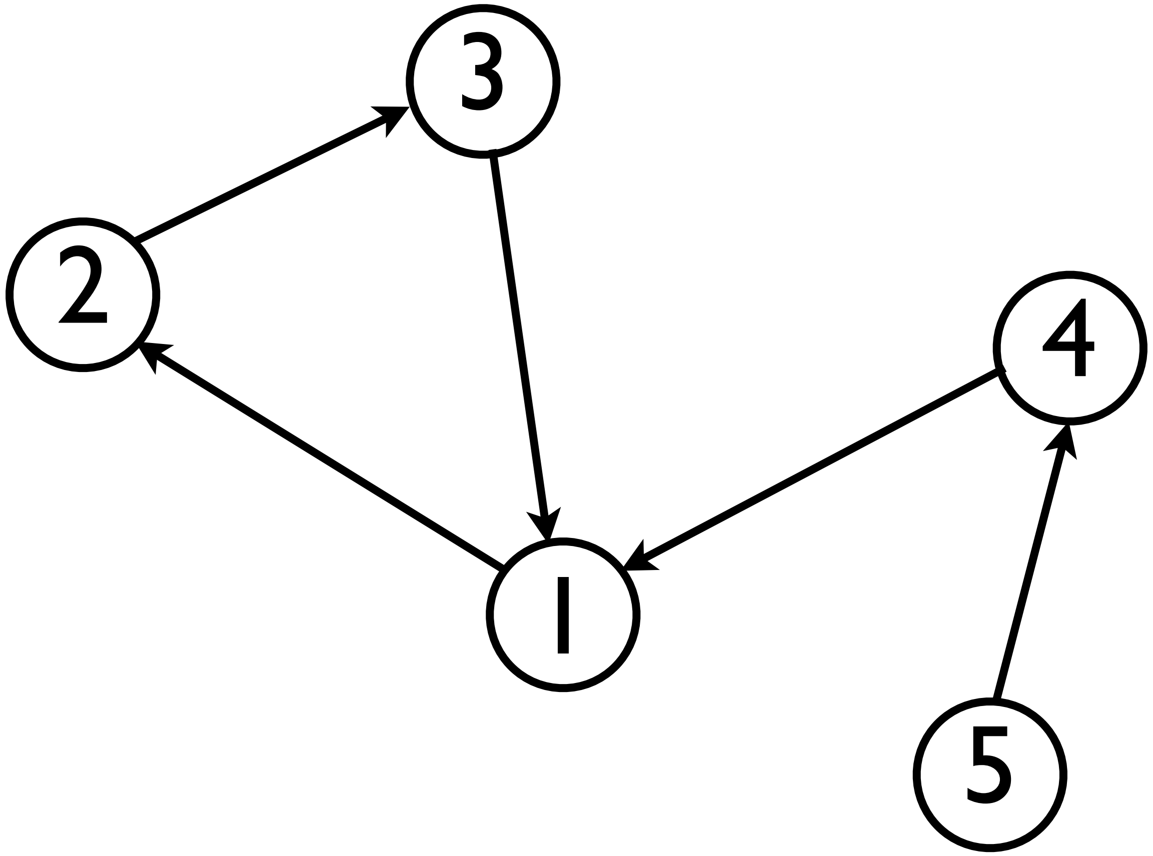
- To represent the set of edges, we can use two alternative representations:
  - An **adjacency matrix** *A* for the whole graph.
  - An **adjacency list** for every node $m \in N$.

Adjacency matrices

- An **adjacency matrix** *A* is an |*N*| x |*N*| matrix, where |*N*| is the number of nodes in the graph.
    - For an *unweighted* graph, the (*mn*)th entry of A contains a 1 or a 0 depending on whether edge $(m, n) \in E$.
    - For a *weighted* graph, the (*mn*)th entry of A contains the *weight* of edge $(m, n) \in E$.
        - If $(m, n) \notin E$, then we can store either 0, infinity, or null (depending on what's most useful).

Adjacency matrices

Example for *directed* graph:



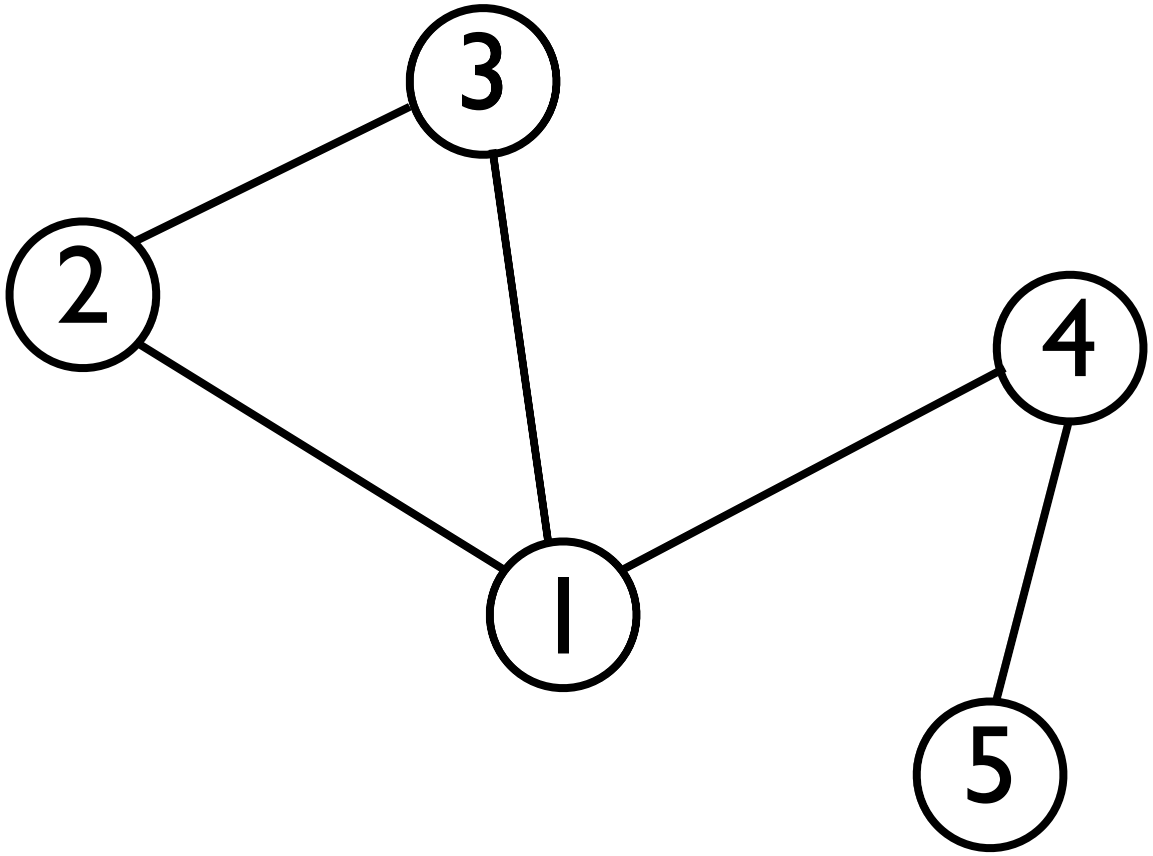
$$\begin{array}{cc|c|c|c|c|c|}
 & & & & n & & \\
 & & 1 & 2 & 3 & 4 & 5 \\
\hline
 & 1 & & 1 & & & \\
\hline
 & 2 & & & 1 & & \\
\hline
m & 3 & 1 & & & & \\
\hline
 & 4 & 1 & & & & \\
\hline
 & 5 & & & & 1 & \\
\hline
\end{array}$$

Adjacency matrices

Example for *undirected* graph:

In an *undirected* graph, the adjacency matrix A equals its own transpose (i.e., $A = A^{\mathsf{T}}$).

$$
\begin{array}{cc|c|c|c|c|c|}
 & & & & n & & \\
 & & 1 & 2 & 3 & 4 & 5 \\ \hline
 & 1 & & 1 & 1 & 1 & \\ \hline
 & 2 & 1 & & 1 & & \\ \hline
m & 3 & 1 & 1 & & & \\ \hline
 & 4 & 1 & & & & 1 \\ \hline
 & 5 & & & & 1 & \\ \hline
\end{array}
$$

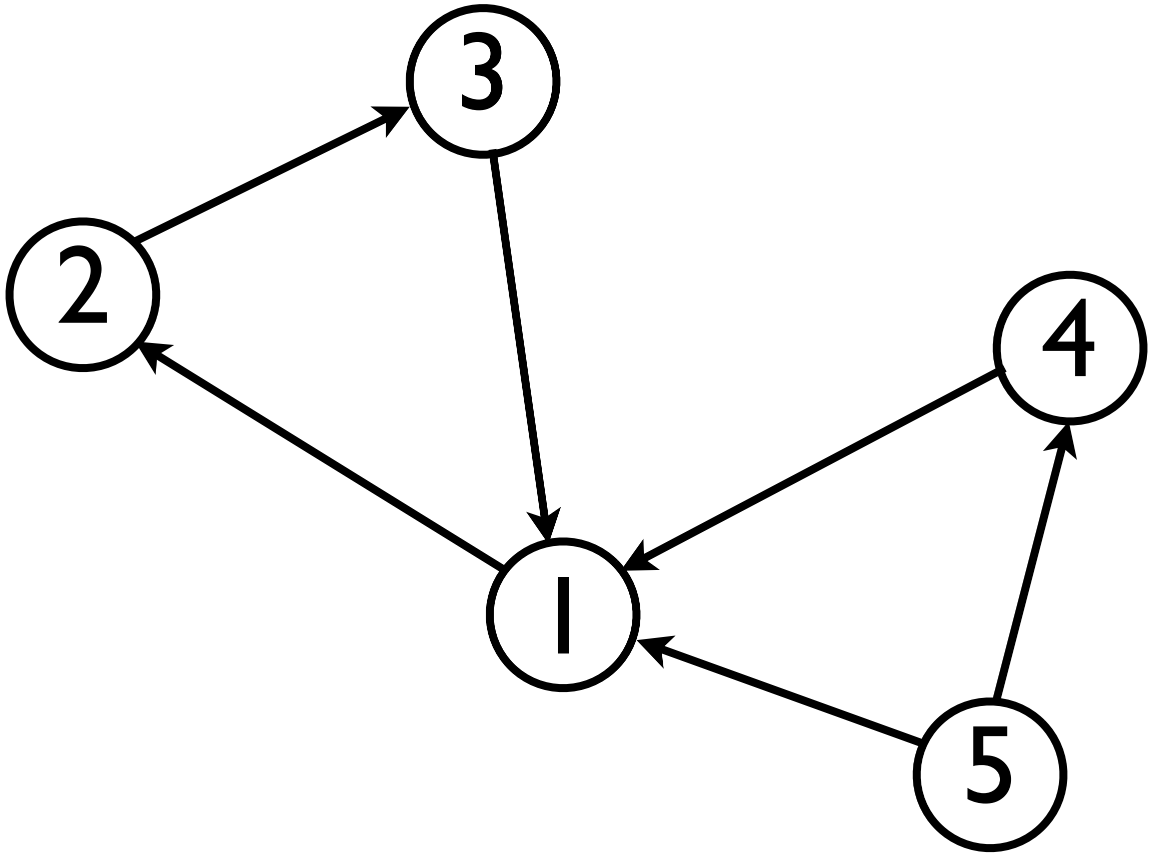


Adjacency matrices

- Adjacency matrices offer *fast access* to the presence/absence of any edge in the graph.
- However, for graphs in which edges are *sparse*, they are space-inefficient ($O(|N|^2)$).
- A space-saving (but slower) alternative is adjacency lists...

Adjacency lists

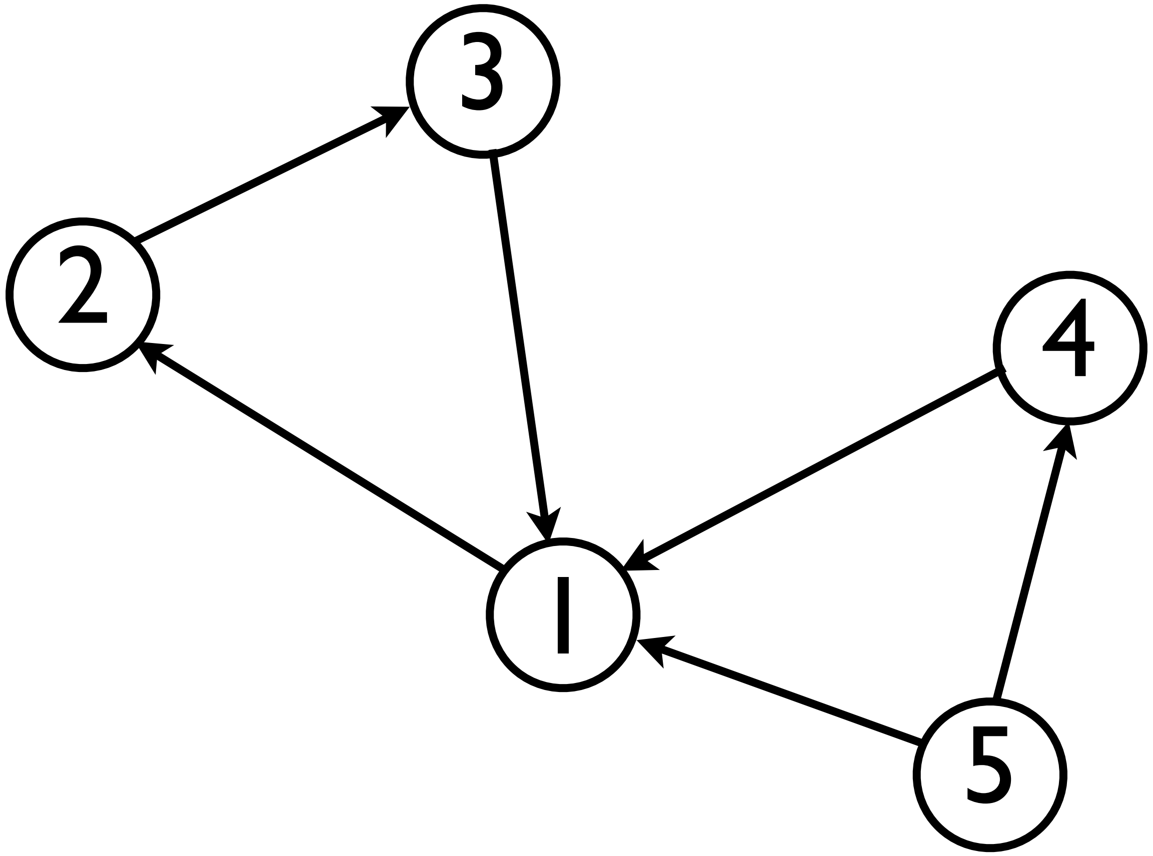
- With adjacency lists, every node maintains a *list* of other nodes to which it is connected.



Node 1: { 2 }
Node 2: { 3 }
Node 3: { 1 }
Node 4: { 1 }
Node 5: { 4, 1 }

Adjacency lists

- Adjacency lists require only *O*(|*E*|) space to store all the edges.
- However, they require *O*(|*E*|) time to *find* a particular edge.



Node 1: { 2 }
Node 2: { 3 }
Node 3: { 1 }
Node 4: { 1 }
Node 5: { 4, 1 }

Graphs in computer science

- Graphs find many uses in computer science in almost every sub-discipline:
  - Computability/complexity theory.
  - Networking.
  - Machine learning.
  - Social networks.
  - Compilers
  - ...

Graphs in computer science

- Here, we will give a very superficial (but hopefully better than no) treatment of graphs.
- One of the fundamental algorithms associated with graphs is finding the *shortest path* between any two nodes *m*, *n*.
- This has applications in many real-world problems, such as...

Kevin Bacon and Erdős numbers

- ...